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Predicting Risk with Risk Measures: An Empirical Study

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Abstract

In this study we consider the risk estimation as a stochastic process based on the Sample Quantile Process (SQP) - which is a generalization of the Value-at-Risk calculated on a rolling sample. Using SQP's, we are able to show and quantify the pro-cyclicality of the current way financial institutions measure their risk. Analysing 11 stock indices, we show that, if the past volatility is low, the historical computation of the risk measure underestimates the future risk, while in periods of high volatility, the risk measure overestimates the risk.

Moreover, using a simple GARCH(1,1) model, we conclude that this pro-cyclical effect is related to the clustering of volatility. We argue that this has important consequences for the regulation in times of crisis.

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J*EL classification*: C13; C22; C52; C53; G01; G32

Keywords: risk measure; sample quantile process; stochastic model; VaR; volatility

1 Introduction

The introduction of risk based solvency regulations has brought the need for financial institutions to evaluate their risk on the basis of probabilistic models. Since the riskmetrics attempt by JP Morgan, the capital needed by companies to cover their risk is often identified to the quantile at a certain threshold α of the return distribution of the portfolio (see [19]). They called this risk measure Value-at-Risk (VaR), a name that has been adopted by the financial community. The question of the appropriateness of the risk measure to use for evaluating the risk of financial institutions has been heavily debated since the financial crisis of 2008/2009. For a review of the arguments on this subject, we refer e.g. to [11].

Apart from the choice of adequate risk measures, there is an accepted idea that risk measurements are pro-cyclical: in times of crisis, they overestimate the future risk, while they underestimate it in quiet times. The general issue of procyclicality has been analyzed with respect to the implications for banking regulation. The discussions have mostly focused on credit risk measurement; see e.g. [13, 12, 17] for contributions with respect to Basel II or, with respect to Basel III, [16, 3]. Consequently, this issue has been adressed explicitly in Basel III by creating a so called 'counter-cyclical' capital buffer (see [4]). Still, as in Basel III, most of the analyses focus on a macro-economic perspective. There have been little attempts to study the direct relation between risk measure estimates and procyclicality empirically (see e.g. [5]), and even less to quantify it. Some studies looked at the effect of crisis on the VaR estimation (see [15]) or on new ways of estimating VaR itself (see [18]).

A full quantification of procyclicality requires a dynamic reformulation of the risk measurement departing from a pure static approach estimating quantiles of an unconditional distribution. To do so, in this paper, we use an approach which generalizes in a simple way the static 'regulatory' risk measure VaR to a dynamic one. We consider the measurement itself as a random process, introducing the sample quantile process (SQP) first proposed in [14] and developed in [1, 10] as a way to measure risk. Using the SQP, we explore its dynamic behavior and its ability of predicting correctly the future risk. This allows us to precisely quantify empirically the procyclicality through a new way of studying risk measurements. Its understanding is central to an effective risk management and to the discussion which risk measures to use.

We conduct our study on 11 stock indices (SI) of major economies to analyze the common behaviors and use them as realizations of the SQP process that we are exploring. When integrating with respect to the Lebesgue measure in the definition of SQP, it collapses to the evaluation of the sample VaR, with rolling windows. We explore then various SQPs based on a random measure chosen as the absolute value of the log-return taken at a power p.

We assume that the sample measurement is an estimation of the future risk of the time series, as commonly done in Solvency regulations. We define a look-forward ratio which quantifies the difference between the historically predicted risk (via the SQP) and the estimated realized future risk (measured a posteriori by the VaR realized one year later). We use the ratio to quantify the accuracy of risk estimation. We run the empirical analysis over the different stock indices and find that the ratio is on average above 1 but with a high (between .4 and .5!) Root Mean Square Error (RMSE), which means large fluctuations above and below 1.

Since the volatility is high in crisis time and low when there is little information hitting the market, it can be used as a proxy for the market state. That is why we suggest an analysis of the look-forward SQP ratio conditioned to the realized volatility, as done in [8] for the autocorrelation of the log-returns. We find a relatively strong (between 49% and 58% for $p = 0,\ 0.5$) negative linear correlation between the log-ratio and the volatility, as well as a negative rank correlation (between 45% and 50% for the Spearman ρ, again for $p = 0,\ 0.5$) We find that in times of high volatility, the ratio is significantly below 1, while it can go even over 3 in times of low volatility signaling a strong underestimation of future risk.

Looking for explanations of this effect, we model the stock indices log-returns by GARCH(1,1), a process that models the volatility clustering present in the data. Note that we use on purpose a simple GARCH to isolate the volatility clustering effect, and do not claim that this process describes well financial asset returns; indeed we do not account well for the fatness of the tails, taking a normal innovation instead of a Student one. The calibrated GARCHs are stationary and reproduce very well the average volatility. Using Monte Carlo simulations, we show that GARCH presents a very similar behavior and gives also a strong negative correlation between the log-ratio and the volatility. This negative correlation is stronger than that obtained on the data (the strength might be a bit oversized due to the normal innovation). We conclude that the procyclicality is related to the clustering and the return to the mean of the volatility.

The paper is organized as follows. In Section 2, we formally introduce the Sample Quantile Processes, discuss their construction and explore their empirical behavior with 11 stock indices. In Section 3, we explore the ability of the SQP's to correctly predict the risk for the future year. By conditioning them on the annual volatility in Section 4, we show that they underestimate the risk in times of low volatility, while overestimating it in times of high volatility, confirming the pro-cyclicality of risk measurements. Introducing the GARCH(1,1) process in Section 5, we show that it can reproduce these effects well. In Section 6, we summarize the results and discuss their consequences and future developments.

2 Risk Measure as a Stochastic Process

Since the risk is dynamic, changing with time, we choose a risk measure that is a stochastic process. We use the simplest dynamic extension of the VaR, namely the Sample Quantile Process, first introduced in [14], then studied in [1, 10].

2.1 Definitions

First, let us quickly recall the definition of the VaR. For a loss random variable L having a continuous distribution function F_L, the VaR at level α of L is simply the quantile of order α of L:

$$VaR_\alpha(L) = \inf\Big\{x : P(L > x) \leq 1-\alpha\Big\} = F_L^{-1}(\alpha) \tag{2.1}$$

where F_L^{-1} denotes the generalized inverse function of F_L. The VaR is generally estimated on historical data, using the empirical quantile $F_n^{-1}(\alpha)$ associated to a n-loss sample $(L_1,\ldots,L_n)$ with $\alpha \in\ (0,1)$ defined by:

$$F_{n;L}^{-1}(\alpha) = \inf\Big\{x : \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(L_i \leq x)} \geq \alpha\Big\}. \tag{2.2}$$

Note that our choice of definition (2.2) is such that the VaR equals an order statistic of the data. It would also be possible to obtain the VaR by linear interpolation between two order statistics. This might provide a slightly different value: the smaller the data set and/or the higher the threshold, the more the two quantile definitions usually differ. By construction and given our choice of definition of a quantile, the empirical quantile will then be larger or equal to that obtained by linear interpolation. Let us turn to the Sample Quantile Process (SQP). The loss is now represented by a stochastic process $L = (L_t)_t$. The SQP $(Q_{\mu,\alpha,T,t}(L))_{t\geq 0}$ of L at threshold α, with respect to a random measure μ defined on $\mathbb{R}^+$, is defined, for $0 \leq T < t$, by (see e.g. [1, 10]):

$$Q_{\mu,\alpha,T,t}(L) = \inf\left\{x : \frac{1}{\int_{t-T}^{t}\mu(s)ds}\int_{t-T}^{t} \mathbb{1}_{(L_s\leq x)}\mu(s)ds \geq \alpha\right\}. \tag{2.3}$$

The SQP, denoted also by $(Q_{\mu,\alpha,T}(t))_{t\geq 0} = (Q_{\mu,\alpha,T,t})_{t\geq 0}$ when no confusion is possible, is clearly a dynamic generalization of the VaR. Indeed, choosing μ in (2.3) as the Lebesgue measure, corresponds to a rolling window VaR denoted by

$$Q_{0,\alpha,T}(t) = \inf\left\{x : \frac{1}{T}\int_{t-T}^{t} \mathbb{1}_{(L_s\leq x)}ds \geq \alpha\right\}. \tag{2.4}$$

This can be read as the continuous version of the VaR defined in (2.2) on the interval $[t-T,t)$. Nevertheless, in practice, we estimate $Q_{0,\alpha,T}(t)$ for discrete observations L_i of L, which comes back to considering a 'rolling window VaR'.

Our approach is to explore the SQP as a predictor of the risk, considering μ as a random measure. We focus on the particular case where μ is defined by $\mu(s) = \mu_p(s) = |L_s|^p$, with $p \in \mathbb{N}$:

$$Q_{p,\alpha,T,t} = Q_{p,\alpha,T}(t) = \inf\left\{x : \frac{1}{\int_{t-T}^{t}|L_s|^p ds}\int_{t-T}^{t} \mathbb{1}_{(L_s\leq x)}|L_s|^p ds \geq \alpha\right\}. \tag{2.5}$$

This choice of μ is arbitrary, but it is one of the simplest formulations that includes the loss, and gives back the Lebesgue measure when taking $p = 0$ (i.e. the VaR process with a rolling window). It is also a way to study the impact of extreme movements: the higher p, the more emphasis we put on the extremes. Looking at (2.5), we can see that positive values of p will let the SQP put more weight on the tail of the distribution. And, correspondingly, negative values of p will let the SQP put more weight on the center of the distribution. We concentrate on non-negative values of p as we are interested in capturing and modelling the tail risk.

Throughout the paper, we consider the empirical estimator $(\widehat{Q}_{p,\alpha,T,t})_{t\geq 0}$ of the SQP defined by

$$\widehat{Q}_{p,\alpha,T,t} = \inf\left\{ x : \frac{1}{\sum_{i\in[t-T,t)}|L_i|^p} \sum_{i\in[t-T,t)} \mathbb{1}_{(L_i\leq x)}|L_i|^p \geq \alpha \right\}. \tag{2.6}$$

2.2 Empirical exploration of the SQP

Given this formalism, we are now going to empirically build realized time series of SQP's computed over various stock indices, which we consider as various sample paths of the process. Since our goal is to look at the appropriateness of the VaR calculations from financial institutions, it seems natural to use stock indices and see how our various SQP's perform on this data.

2.2.1 Data

We consider data from 11 different stock indices. The data used are the daily closing prices from Friday, January 2, 1987 to Friday, December 30, 2016. In Table 1 you can find detailed information about the countries and indices used. The reason for such a large set of data is the need to see if the process has similar behavior over most of the developed economies. Moreover, we want a relatively long dataset, which is offered by all these stock indices.

Table 1: Information about the different indices used

Country	Country Code	Index	Description
Australia	AUS	AORDD	The All Ordinaries index (AOI) is made up of the 500 largest companies listed on the Australian Securities Exchange (ASI)
Canada	CAN	GSPTSED	The Canada S&P/TSX 300 Composite comprises approximately 71% of market capitalization for Canadian-based, Toronto Stock Exchange (TSX) listed companies
France	FRA	CACTD	The CAC All-Tradable Index represents the 250 largest stocks, by capitalisation, traded on Euronext Paris
Germany	DEU	CDAXD	Germany CDAX Total Return Index includes the shares of all companies listed on Frankfurt Stock Exchange
Great Britain	GBR	FTASD	UK FTSE All-Share Index is capitalisation-weighted and comprises more than 600 (out of 2,000+) companies traded on the London Stock Exchange
Italy	ITA	BCIPRD	Italy BCI Global Return Index includes over 325 companies listed on the Milan Stock Exchange
Japan	JPN	TOPXD	The Tokyo Price Index (TOPIX) is a capitalization-weighted price index of all First Section stocks (more than 1600 companies)
Netherlands	NLD	AEX	The AEX index is a stock market index composed of a maximum of 25 of the most actively traded securities of Dutch companies on Euronext Amsterdam
Singapore	SGP	STI	FTSE Straits Times Index (STI) is capitalisation-weighted and tracks the performance of the top 30 companies listed on the Singapore Exchange
Sweden	SWE	OMXAFGX	Sweden OMX Affärsvärldens General Index includes all the shares of companies registered on the Stockholm Stock Exchange
United States of America	USA	SPXD	S&P 500 Composite Price Index represents about 75% of the stock market's capitalization

As commonly done, we refer to the S&P 500 index as our main example and focus on this specific index whenever showing figures. All the results for the other indices are given in the appendix and, if not commented on otherwise, show the same characteristics.

For any time series, let us denote the closing price at time t by $S(t)$ and by Δt the interval between two consecutive spot prices. We focus on daily intervals. We then define the log-return $X_{\Delta t}(t)$ by:

$$X_{\Delta t}(t) = \ln\left(\frac{S(t)}{S(t-\Delta t)}\right) \tag{2.7}$$

Throughout this paper we work with log-returns. If not specified otherwise, by abuse of notation, we will refer to $X_i = X_{\Delta t}(t_i)$ as the daily log-return (with $\Delta t = 1$ day) at time t_i.

Also, in some situations it might be of interest to work with the log-return losses instead of the log-returns, simply defined as: $L_i = -X_i$, the daily log-return loss at time t_i.

2.2.2 Empirical Study and Discussion

At first, we focus on a rather qualitative evaluation, illustrated on the S&P 500. In Figure 1, we plot the sample paths of SQP at level α at monthly frequency and the VaR benchmark on the overall sample at the same level α. We look at it for different parameter values p of our random measure $\mu = |L_S|^p$: $p = 0, 0.5, 1, 2$. The sample size, T, is taken to be one year and we consider thresholds $\alpha = 95\%$ and 99%, respectively.

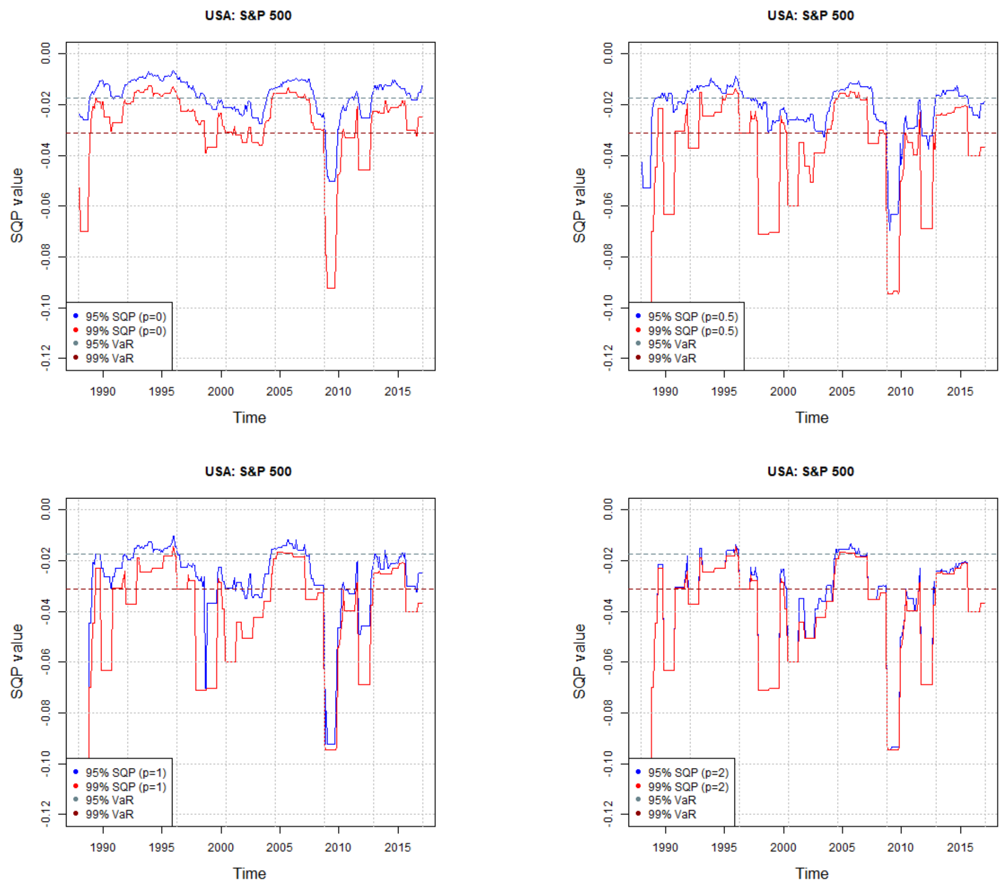

Figure 1: Sample Quantile Processes (SQP) rolling every month with $T = 1$ year, and thresholds $\alpha = 95\%$ and 99%, respectively. From left to right and from top to bottom: $p = 0$, $p = 0.5$, $p = 1$ and $p = 2$. Dashed lines correspond to VaR(α) computed over the whole sample. The y-scale is the same in the four graphs to allow comparison between the different measures.

When $p = 0.5$, the values of the SQP are more volatile (*i.e.* with a wider range) than for $p = 0$. They are not only more volatile but also smaller than the SQP for $p = 0$ (*i.e.* the rolling-window VaR), in particular with extreme values that seem to have decreased notably. Hence, the SQP, with $p = 0.5$, is a measure that enables a larger discrimination between the losses.

For $p = 1$, the properties found for $p = 0.5$ are also present and accentuated. The volatility of the values of the SQP has once again increased for $\alpha = 0.95$. The range of the SQP is also bigger when $p = 1$ than when p is equal to 0 and 0.5. As for $p = 0.5$, the SQP for $p = 1$ is a measure that enables a bigger discrimination between the losses. But we can also see that the differences between the SQP for the two thresholds has clearly diminished for extreme values. To conclude, the SQP for $p = 1$ is close to the SQP for $p = 0.5$, apart for the most extreme values at threshold $\alpha = 0.95$, and continues the trend seen for the latter.

In addition, the SQP with $p = 2$ is very close to the previous SQP with $p = 1$. However, the differences between the SQP with the different thresholds are extremely small and almost vanished. Hence, it is difficult, if not impossible, to distinguish between the 95% and 99% thresholds. The 95% threshold collapses in fact on to the 99% threshold. As can be seen on the last graph, both SQPs hover around the VaR(99%). Moreover, for the 99% threshold, the SQP with $p = 2$ is equal the SQP with $p = 1$, which does not make it interesting anymore. We then conclude that the choice of $p = 1, 2$ to define the SQP may not be the best one to assess the intensity of losses.

In summary, we can observe the following. For a fixed threshold α, the SQP decreases with higher powers of p. Also, the higher the parameter p, the more volatile the SQP gets. At the same time the difference of the SQPs at level $\alpha = 95\%$ and 99% diminishes with increasing p - we could observe that for $\alpha = 99\%$ the SQP for p=1 and p=2 coincide. From the figures, $p = 1/2$ seems optimal for tail discrimination.

These observations can also be confirmed when looking at it more quantitatively as done in Table 2 and its figure for the S&P 500, as well as in Table 3. In Table 2 we compare the average values achieved by the various monthly rolling SQP's with sample size T equal to one year as a function of the parameter p and the threshold α for the S&P 500. We notice from the figure again that for the highest thresholds (above and including 99%), the values for $p = 1$ and $p = 2$ are the same for all indices. We see in Table 3 that there is no marked difference between the various indices. The behavior is the same, the higher the value of p, the larger the average (in absolute terms). In the same table, the ratio between the two α is the largest on average for $p = 0.5$ confirming our remarks above about tail discrimination. This is not true for all indices particularly the European ones: FRA, DEU, NLD, GBR do not follow this behavior.

Risk measures	α=95%	α=99%	Ratio SQP(99%)/SQP(95%)
SQP (p=0)	-1.70%	-2.86%	1.68
SQP (p=0.5)	**-2.23%**	**-4.44%**	**1.95**
SQP (p=1.0)	-3.28%	-4.65%	1.37
SQP (p=2.0)	-4.36%	-4.65%	1.03

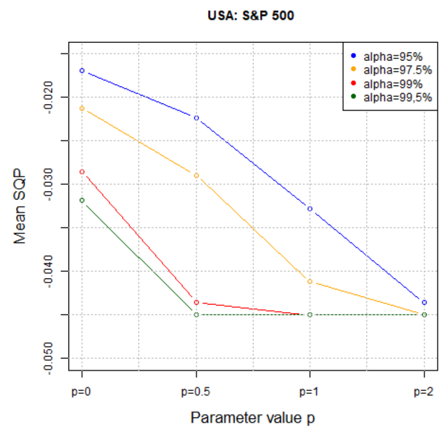

Table 2: Average over the whole sample of rolling-window SQPs with sample size $T = 1y$ (S&P 500) as a function of the power p for various thresholds α.

Table 3: Average over the whole sample of the rolling-window SQPs with sample size $T = 1y$ for each of the 11 stock indices (in %) and associated ratio of the SQPs between two thresholds (95 and 99%) - using historical data. In the last column, we present the average over all indices ± the standard deviation.

Mean for	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG (± σ)**
						Mean 1y in % (α = 95%)						
$p = 0$	-1.47	-1.47	-1.96	-1.96	-2.11	-2.01	-2.05	-1.81	-2.05	-1.61	-1.70	**-1.84 (0.24)**
$p = 0.5$	-1.96	-2.00	-2.62	-2.61	-2.74	-2.73	-2.79	-2.70	-2.71	-2.16	-2.23	**-2.48 (0.32)**
$p = 1$	-3.24	-2.75	-3.21	-3.60	-3.55	-3.86	-3.47	-4.20	-3.39	-2.78	-3.28	**-3.39 (0.42)**
$p = 2$	-4.20	-3.76	-4.41	-4.40	-4.77	-5.27	-4.57	-5.47	-4.69	-3.63	-4.36	**-4.50 (0.55)**
						Mean 1y in % (α = 99%)						
$p = 0$	-2.41	-2.58	-3.25	-3.23	-3.46	-3.42	-3.46	-3.45	-3.32	-2.75	-2.86	**-3.11 (0.39)**
$p = 0.5$	-4.15	-3.69	-4.33	-4.28	-4.73	-5.26	-4.45	-5.38	-4.61	-3.55	-4.35	**-4.44 (0.56)**
$p = 1$	-4.33	-3.85	-4.55	-4.59	-4.98	-5.42	-4.76	-5.60	-4.90	-3.73	-4.50	**-4.65 (0.57)**
$p = 2$	-4.33	-3.85	-4.55	-4.59	-4.98	-5.42	-4.76	-5.60	-4.90	-3.73	-4.50	**-4.65 (0.57)**
						Ratio of Mean 1y at 99% over 95%						
$p = 0$	*1.64*	*1.76*	*1.66*	*1.65*	*1.64*	*1.70*	*1.69*	*1.90*	*1.62*	*1.71*	*1.68*	***1.70 (0.08)***
$p = 0.5$	*2.12*	*1.85*	*1.65*	*1.64*	*1.73*	*1.93*	*1.60*	*2.00*	*1.70*	*1.64*	*1.95*	***1.80 (0.18)***
$p = 1$	*1.33*	*1.40*	*1.42*	*1.28*	*1.40*	*1.40*	*1.37*	*1.34*	*1.45*	*1.34*	*1.37*	***1.37 (0.05)***
$p = 2$	*1.03*	*1.02*	*1.03*	*1.04*	*1.04*	*1.03*	*1.04*	*1.02*	*1.04*	*1.03*	*1.03*	***1.03 (0.01)***

3 Predictive Power for the Risk using the SQP

After having introduced and analyzed the SQP we now want to focus on our main goal to test the SQP's ability to assess the estimated future risk correctly - or what we call predictive power. Let us briefly point out why this differs from the classical backtesting, which designates a statistical procedure that compares realizations with forecasts. First recall that, for the VaR, some of the backtesting methods used by European banks and the Basel committee are based on the so-called violation process counting how many times the estimated/forecast VaR(α) has been violated by realized losses during the following year; those VaR 'violations' should form a sequence of iid Bernoulli variables with probability $(1-\alpha)$ (see e.g. [6, 7]).

We want to assess the risk, thus the quality of estimation. While the violation ratio in [9] gives a measure of the quality of the estimation, here we do not question at all if the violation process is in line with the acceptable number of exceptions. Our goal is rather, when considering the risk estimation as a stochastic process, to quantify how well or wrong our VaR estimate performed, compared to the realized future value of the VaR. It is an alternative and new way of looking at risk, which, subsequently, leads also to another way to assess the quality of the risk estimation. Moreover, this way of risk quantification can be applied to any other risk measure besides the VaR (as for instance to the Expected Shortfall).

3.1 Look-forward SQP Ratio

For this, we consider SQP estimators $(\widehat{Q}_{p,\alpha,T}(t))_t$, computed according to (2.6) during a period (of length T years) that ends at time t; for ease of notation, we may omit the hat in the following.

We measure the quality of the risk estimation by comparing it to the VaR estimated one year later ($p=0, T=1$), i.e. $Q_{0,\alpha,1}(t+1y)$. It means to check if our estimate $Q_{p,\alpha,T}(t)$ is a good estimate at time t of the future and unknown value $Q_{0,\alpha,1}(t+1y)$. Hereby, it is important to note that the reference value, the VaR one year later $Q_{0,\alpha,1}(t+1y)$, will always be computed over a period of one year ($T=1$) as it should represent the realized risk during that period of time. Note also that we compare all versions of the SQPs to the VaR as this is the reference risk measure asked by many regulators.

This leads us to introducing a new random variable, the ratio of SQPs denoted by $R_{p,\alpha,T}(t)$, that quantifies the difference between the historically predicted risk $Q_{p,\alpha,T}(t)$ and the estimated realized future risk $Q_{0,\alpha,1}(t+1y)$ (considered at the time t plus one year later) computed using the period of time $[t, t+1y)$:

$$R_{p,\alpha,T}(t) = \frac{Q_{0,\alpha,1}(t+1y)}{Q_{p,\alpha,T}(t)}. \tag{3.1}$$

Table 4: SQP ratios (defined in (3.1)) average over the whole historical sample, for 1 year, for each index and for two different thresholds (95 and 99 %). In the last column, we present the average over all indices ± the standard deviation.

Mean for	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG ($\pm\ \sigma$)**
	SQP ratio for 1y ($\alpha = 95\%$)											
$p = 0$	1.04	1.09	1.08	1.11	1.10	1.11	1.10	1.06	1.10	1.08	1.06	**1.08 (0.02)**
$p = 0.5$	0.81	0.83	0.81	0.85	0.84	0.85	0.83	0.78	0.84	0.83	0.82	**0.83 (0.02)**
$p = 1$	0.64	0.65	0.65	0.64	0.66	0.68	0.66	0.60	0.68	0.68	0.67	**0.66 (0.02)**
$p = 2$	0.50	0.47	0.50	0.51	0.49	0.50	0.50	0.44	0.50	0.51	0.50	**0.49 (0.02)**
	SQP ratio for 1y ($\alpha = 99\%$)											
$p = 0$	1.04	1.10	1.06	1.10	1.11	1.14	1.11	1.12	1.09	1.08	1.08	**1.09 (0.03)**
$p = 0.5$	0.80	0.81	0.85	0.88	0.83	0.85	0.88	0.82	0.83	0.88	0.84	**0.84 (0.03)**
$p = 1$	0.74	0.77	0.79	0.80	0.78	0.79	0.80	0.74	0.76	0.81	0.79	**0.78 (0.02)**
$p = 2$	0.74	0.77	0.79	0.80	0.78	0.79	0.80	0.74	0.76	0.81	0.79	**0.78 (0.02)**

To be clear, let us look at a simple example. Choosing $\alpha = 0.95$, $t =$ January 2014, and $T = 1$ year, the denominator $Q_{p,0.95,1}$ of the ratio $R_{p,0.95,1}(t)$ corresponds then to the empirical estimate of the SQP computed on a 1 year sample from January 2013 ($t - T$) to December 2013 (*i.e.* the month before t, as we are considering the interval $[t - T, t)$), which acts as an estimate for our future risk in 2014 (if T would be 3 years, $Q_{p,0.95,3}$ would be computed from January 2011 up to December 2013). We define the future risk $Q_{0,\alpha,1}(t + 1y)$ as the VaR estimated on the one year sample from January 2014 (t) until December 2014 (last month before $t + 1y$). Finally, the ratio of the two quantities is built.

Consequently, if we look at the values of the ratio $R_{p,\alpha,T}(t)$, a value near 1 means the prediction $Q_{p,\alpha,T}(t)$ correctly assesses the future risk estimated by $Q_{0,\alpha,1}(t + 1y)$. Similarly, if the ratio is above one, we can say that we have an under-estimation of the future risk, i.e. the observed future risk. And to the contrary, if it is below one, we have an over-estimation of the future risk, which would lead to an over-evaluation of the needed capital.

3.2 Assessing the average SQP ratio

Taking a monthly rolling window for the ratios $R_{p,\alpha,T}(t)$ (i.e. updating t every month), we compute the average SQP ratio over the whole sample. Here we focus on looking at the overall averages (over the 11 different indices) of the monthly SQP ratios. Results for the 11 different indices on their own are reported in Table 4.

We see, in Table 4, that all the ratios are clearly unequal to one, telling us that the future risk is, on average, not well predicted. Furthermore, we clearly see for both choices of α that the average ratio is above 1 for $p = 0$ - even after two standard deviations. We find back the well known result that the historical estimation of VaR underestimates the risk. For $p > 0$, we see that the values are below 1 and clearly decrease with p for both $\alpha = 95\%$ and 99% (apart from $p = 2$ for $\alpha = 99\%$). This implies that our choice of parameter p puts too much weight in the tails on average, causing a clear overestimation of the actual risk. When comparing the average ratio for our two thresholds α, we see that it does not change a lot for $p = 0, 0.5$, while the ratio increases with α for $p = 1, 2$. This increase is not surprising as for those values of p, more weight is added in the tail, so the larger the threshold α, the smaller the difference to the actual future risk, and thus the smaller the overestimation.

Clearly, only considering the average does not tell the whole story as it does not account for the dynamics of the rolling window - after all, we are computing the ratios on a monthly basis. Therefore, in a second step we choose to look at the Root Mean Square (RMS) distance and compute the associated Root Mean Square Error (RMSE), i.e. the average RMS distance of our ratio from a perfect prediction which equals a ratio of 1:

$$RMSE := \sqrt{\frac{1}{N}\sum_{t=1}^{N}(R_{p,\alpha,T}(t) - 1)^2} \tag{3.2}$$

where N is the number of overall time points the ratio is evaluated at (as we are working with a monthly rolling window, N equals 12 times the amount of years on which the ratio is tested on). This RMSE measures the "error" of the forecasted risk for a certain sample path.

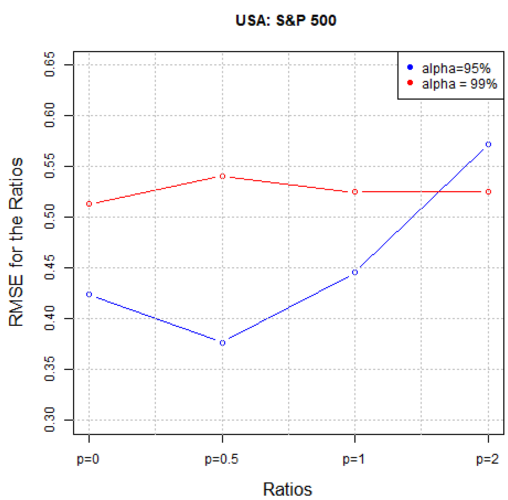

Figure 2: RMSE (defined in (3.2)) of the SQP Ratios $R_{p,\alpha,1}(t)$ taking $p = 0, 0.5, 1, 2$ and two thresholds $\alpha = 95\%$ (in blue), 99% (in red), on the S&P 500

For the S&P 500 index, we see in Figure 2 that the error increases with the threshold: it is smaller for 95% than for 99%, except for the SQP measure with p=2. For the 99% threshold, the error increases slightly from $p = 0$ to $p = 0.5$ where it has its maximum, and levels slightly off to its minimum for $p = 1, 2$. We observe a different, almost opposite pattern in the case of 95%: it decreases from $p = 0$ to its minimum for $p = 0.5$, then increases with p. All the values are clearly above 0, suggesting that the various measures do not forecast well the future risk.

As before, when considering the average SQP Ratio, we observe a similar - although slightly different - behavior for the average of all indices as with the S&P 500 index (see the last two

Table 5: Average RMSE for the SQP ratios (eq. 3.1, with $T = 1$ year) - using historical data for each index. In the last column, we present the average over all indices $\pm$ the standard deviation.

Mean for	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG** ($\pm$ σ)
	RMSE of the SQP ratio for 1y ($\alpha = 95\%$)											
$p = 0$	0.36	0.46	0.44	0.50	0.49	0.54	0.52	0.46	0.47	0.44	0.42	**0.46 (0.05)**
$p = 0.5$	0.33	0.42	0.38	0.43	0.39	0.46	0.42	0.42	0.39	0.37	0.38	**0.40 (0.03)**
$p = 1$	0.44	0.47	0.43	0.47	0.45	0.50	0.46	0.52	0.44	0.42	0.45	**0.46 (0.03)**
$p = 2$	0.56	0.59	0.54	0.55	0.55	0.60	0.55	0.62	0.56	0.53	0.57	**0.57 (0.03)**
	RMSE of the SQP ratio for 1y ($\alpha = 99\%$)											
$p = 0$	0.37	0.56	0.39	0.50	0.52	0.61	0.58	0.72	0.52	0.47	0.51	**0.52 (0.10)**
$p = 0.5$	0.45	0.49	0.39	0.45	0.42	0.58	0.48	0.65	0.46	0.44	0.54	**0.49 (0.08)**
$p = 1$	0.45	0.49	0.38	0.42	0.40	0.53	0.44	0.59	0.45	0.41	0.52	**0.46 (0.06)**
$p = 2$	0.45	0.49	0.38	0.42	0.40	0.53	0.44	0.59	0.45	0.41	0.52	**0.46 (0.06)**

columns in Table 5). On average, the error increases with the threshold (except for $p = 1, 2$), and - as with the S&P 500 - for $\alpha = 0.95$ the error is smallest for $p = 0.5$ and increases afterwards again. For $\alpha = 0.99$ its maximum is reached for $p = 0$, then the error levels off, being constant for $p = 1, 2$. Again, the average RMSE in all cases is clearly above 0 reaching 50% of the right value 1. This means that there are high fluctuations of the ratio, for the weak underestimation (of around 10%) for $p = 0$, as well as for a stronger overestimation for $p > 0$ (recall the last column of Table 4). This analysis does not give us yet an explanation for this behavior. We want to understand why and in which circumstances we over- or underestimate the risk measured by the SQP throughout the sample. It is what we tackle in the next sections.

4 Volatility Impact on the Measured Risk

4.1 Realized Volatility as an Indicator of Market States

To better understand the dynamic behavior of the ratios, we condition them on past volatility. This is inspired by the empirical study on Foreign Exchange done by Dacorogna *et al.* in [8], where the authors conditioned the returns on past volatility, showing that, on one hand, after a period of high volatility, the returns tend to be negatively correlated. It means that the volatility should diminish in the future, with a distribution more concentrated around its mean; thus the quantiles should be smaller. It implies that the future risk would be overestimated. On the other hand, in times of low volatility, consecutive returns tend to be positively auto-correlated; thus the volatility should increase on the long term, with the consequence in the future of quantiles shifted to the right, meaning that our estimation would underestimate the future risk.

To be able to condition on volatility, we define an empirically measured volatility, $\hat{\sigma}(t-1)$, also called in the literature 'realized volatility', taken over a rolling sample of size $T = 1$

year (or 3 years) up to but not including a time t, and we annualize it by considering $\hat{\sigma}(t-1) \times \sqrt{252}$, obtaining the annual realized volatility $v(t)$ at time t defined as:

$$v(t) = \hat{\sigma}(t-1) \times \sqrt{252} \quad \text{where} \quad \hat{\sigma}(t-1) := \sqrt{\frac{1}{N-1} \sum_{i=t-N}^{t-1} \left(X_i - \frac{1}{N} \sum_{i=t-N}^{t-1} X_i \right)^2} \tag{4.1}$$

N being the number of days in a sample of size T years, and the daily log-returns X_i being defined in (2.7). Note that the notation is chosen such that the annual realized volatility $v(t)$ at time t (January 1, 2014 for instance) is computed on the previous year until $t-1$ included (i.e. until December 31, 2013, in our previous example). We evaluate $v(t)$ on a rolling basis every month. Hereby, we obtain a time series of annual realized volatility $(v(t))_t$, which can be used as a benchmark of the current market state.

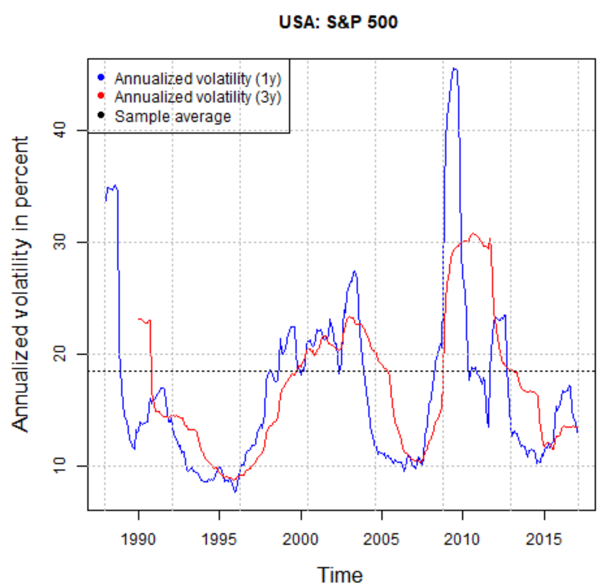

Figure 3

Annual realized volatility (defined in (4.1)) of the S&P 500 returns over a rolling sample of size $T = 1, 3$ year(s). The dashed line represents the sample average of the volatility

In Figure 3, we present $(v(t))_t$ for the S&P 500 (log-)returns between 1987 and 2016 for both a rolling sample of 1 year and of 3 years. We also indicate the average annual realized volatility of the index calculated over the whole sample (represented by the dashed line). Its value is 18.46%. We can see that the volatilities above this benchmark mostly stand for periods of high instability or crisis (and not only in the USA). The high volatility of the period 1987-1989 is for instance explained by the New York Stock Exchange crash in

October 1987. In 1997, Asia was hit by a crisis, as well as Russia in 1998 and Argentina in 1999-2000. In 2001, the United States experienced the bursting of the internet bubble. Following the Lehman Brother's bankruptcy, the period of 2008-2009 was a period of very high volatility. And finally, the sovereign debt crisis in Europe also impacted the S&P 500 Index in 2011-2012. This is an illustration of the increased dependence between the markets during times of crisis.

We conclude from this behavior that our measure of realized volatility is a good measure that reacts well and quickly to various market states. It is thus a reasonable proxy to qualify the times of high risks and to use it to discriminate between quiet periods and periods of crisis. We use it to condition our statistics (the SQP ratio) and see if we can detect different behaviors of the price process (which underlies the risk measure) during these periods in comparison to quiet times, as it was the case for returns in [8]. It is a way to look at the ability of our SQP risk measures to correctly predict the future risks and to capture the dynamics of the market.

4.2 SQP Predictive Power as a Function of Realized Volatility

In the case of the S&P 500 we show in Figure 4 both the annualized volatility and the realized ratios of SQP's for both thresholds 95% and 99%, and for $p = 0$ (rolling VaR). One can easily observe on these graphs a strong opposite behavior for SQP ratios computed on a 1-year sample (in blue) and for volatility (in red). Similar observations hold when using a 3-year sample to predict the estimated future risk (the figures for the analysis of the S&P 500 using a 3-year sample can be found in the appendix).

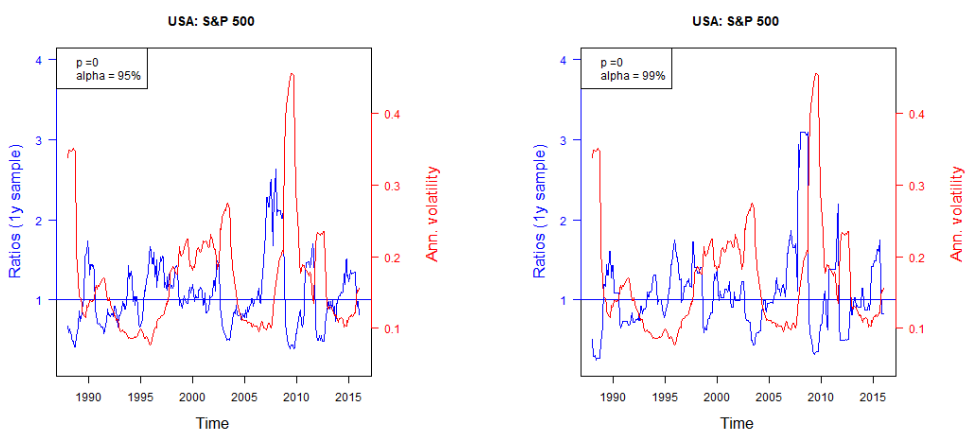

Figure 4: SQP Ratios and annualized volatility time series computed over 1 year ($T = 1y$) for $\alpha = 95\%$ (left) and $\alpha = 99\%$ (right)

In Figure 4 the dynamic behavior appears clearly. The realized ratios can differ from 1 either on the upper side or on the lower side. Looking at the dependence with respect to the annualized volatility, it can be noticed that for both graphs, the realized SQP ratios are negatively correlated with the annualized volatility. This particular feature is of importance in our study because it means that when the volatility in year t is high in the market, the realized SQP ratio in year t is quite low. Recalling the definition of the ratio, this situation means that the risks have been over-evaluated by the SQP calculated in year t (in comparison to the realized risk in year $t+1y$). Conversely, when the volatility is low in year t, the realized SQP ratio is often much higher than 1 in this year, which means that the risks for the year $t+1$ have been under-evaluated by the calculation of the SQP in year t.

Another way of visualizing the negative correlation is to look at the various realized SQP ratios as a function of the annual realized volatility, as in Figure 5.

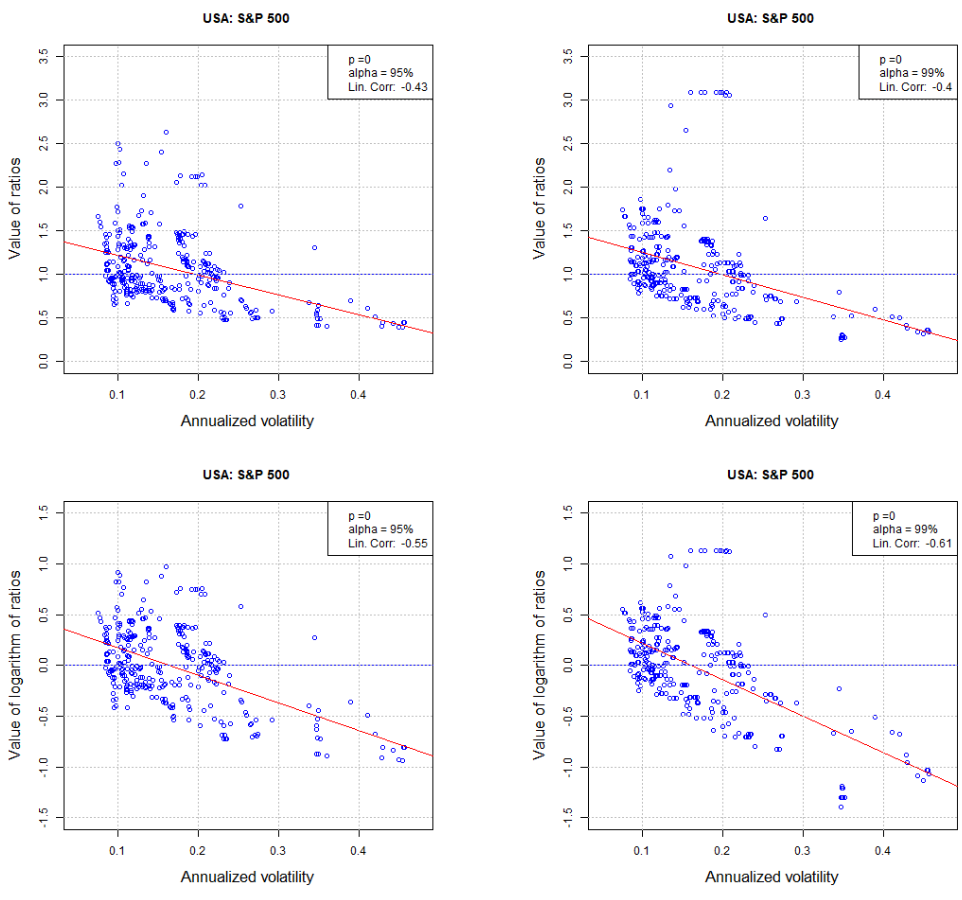

Figure 5: SQP Ratios as a function of annualized volatility for $p = 0$: On the left $\alpha = 95\%$ and on the right $\alpha = 99\%$; first row SQP-ratios, second row logarithm of SQP-ratios

Such figures highlight better the existing negative dependence between these two quantities.

When looking at the first row of Figure 5, the negative correlation between the SQP ratios with the annual realized volatility is clearly noticeable. It is displayed by the negative slope of the simple linear regression line. As it appears also clearly that the (negative) dependence is non linear, we consider the log-ratios instead (see the plots on the second row of Figure 5): the more volatility there is in year t, the lower are the ratios $R_{p,\alpha,T}(t)$. When the log-ratios are negative, it means that losses in year $t+1$ have been overestimated with the measures calculated in year t. In the original scale we can better quantify the magnitude of the effect: When looking at the case of $\alpha = 0.99$, with respect to the overestimation, we can observe ratios much below 0.5, *i.e.* the risk computed at the height of the crisis is more than twice the size of the risk measured a year later. On the other hand, we see that the underestimation can be very big since the realized ratios for the SQP can be even larger than 3; in other words, the risk next year is more than 3 times the risk measured during the current year for this sample. We also observe that the overestimation is very systematic for high volatility, whereas it is not for the underestimation at low volatility. Indeed, in this latter case, we see also values below 1, while we do not see any value above one when the volatility is very high (above 35%). Although we show here only the S&P 500, very similar behaviors are exhibited by all the other indices (see the appendix).

It is of interest to find out which of the various measures μ_p produces the strongest counter-cyclical behavior. To make this more quantifiable, given the highly non-linear behavior of this dependence as illustrated in Figure 5, we choose to look at the linear correlation $\rho(\log(R_{p,\alpha,T}(t), v(t))$ between the *logarithm* of the SQP ratio (instead of the SQP ratio) and the volatility. In Table 6, we present this quantity for all indices and as average over all indices.

Table 6: Pearson correlation between the log of the SQP ratios and the volatility, for each index and for $T = 1$ year, over the whole historical sample, and for two thresholds (95 and 99%). In the last column, we present the average over all indices ± the standard deviation.

	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG (± sd)**
						$\alpha = 95\%$						
p=0	-0.53	-0.52	-0.52	-0.49	-0.48	-0.65	-0.56	-0.51	-0.58	-0.54	-0.55	**-0.54 ± 0.05**
p=0.5	-0.69	-0.52	-0.49	-0.44	-0.40	-0.69	-0.58	-0.63	-0.54	-0.60	-0.57	**-0.56 ± 0.09**
p=1	-0.68	-0.50	-0.42	-0.24	-0.25	-0.51	-0.57	-0.65	-0.42	-0.59	-0.61	**-0.49 ± 0.15**
p=2	-0.67	-0.46	-0.35	-0.17	-0.22	-0.49	-0.43	-0.59	-0.31	-0.50	-0.50	**-0.43 ± 0.15**
						$\alpha = 99\%$						
p=0	-0.66	-0.54	-0.55	-0.49	-0.50	-0.62	-0.60	-0.64	-0.57	-0.57	-0.61	**-0.58 ± 0.05**
p=0.5	-0.66	-0.50	-0.46	-0.29	-0.36	-0.51	-0.49	-0.59	-0.44	-0.52	-0.54	**-0.49 ± 0.10**
p=1	-0.66	-0.51	-0.47	-0.27	-0.39	-0.51	-0.49	-0.60	-0.46	-0.50	-0.54	**-0.49 ± 0.10**
p=2	-0.66	-0.51	-0.47	-0.27	-0.39	-0.51	-0.49	-0.60	-0.46	-0.50	-0.54	**-0.49 ± 0.10**

We see that the Pearson correlation is always significantly negative, the average values for different thresholds and values of p are above 40%. The negative linear correlation is strongest for the lower values of p: in the case of $\alpha = 0.95$ it is strongest for $p = 0.5$ (although $p = 0$ is quite similar), and in the case of $\alpha = 0.99$, it is strongest for $p = 0$. In general one can say that the correlation is similar for both thresholds. One might also conclude that, for those observations, SQP with $p = 0$ or 0.5, are better suited than with higher values of p to emphasize the cyclical signal, thus for designing counter-cyclical models.

Due to the non-linear nature of the dependence, we also consider the notion of rank correlation, looking for instance at the Spearman rho (the values for each index separately as well as for the Kendall tau are available in the appendix). In Table 7, we display only the average ρ over all indices as we have seen in the other tables that indices do not differ significantly from one another in their behaviors. In this table, the results confirm those obtained when considering the correlation between the logarithm of the SQP ratios and the volatility (e.g. highest value in absolute terms for low values of p for both thresholds α, decreasing in p for $\alpha = 0.95$, almost constant in p for $p > 0$ in the case of $\alpha = 0.99$), only the magnitude is a bit lower than for the linear correlation.

Table 7: Average over all indices of the Spearman-ρ between the annualized volatility and the SQP ratios computed on a 1-year sample

	$\alpha = 0.95$	$\alpha = 0.99$
$p = 0$	**-0.47 ± 0.06**	**-0.50 ± 0.04**
$p = 0.5$	**-0.46 ± 0.07**	**-0.45 ± 0.06**
$p = 1$	**-0.42 ± 0.09**	**-0.46 ± 0.06**
$p = 2$	**-0.37 ± 0.11**	**-0.46 ± 0.06**

Note that the same analysis was performed using a 3-year sample (T=3) to compute the estimated future risk $Q_{T,\alpha,t}$ and the annualized realized volatility, then to look at the dependence between annualized volatility and SQP ratios. On average, effects are similar in magnitude and with mostly similar tendencies for both linear correlation and non linear correlation. The corresponding tables can be found in the appendix.

We need to further explore the dependence, to better understand the behavior of the parametrized risk measure (SQP). It is what we develop in the next section, introducing a simple model to highlight the procyclical behavior we observed so far.

5 Looking for Explanations via GARCH modelling

In our empirical analysis, we observe volatility clustering as well as the return to the mean of volatility. Those are both prominent characteristics of a GARCH model. Thus, we think that part of the procyclical behavior with volatility might be explained by this clustering.

Thus, we consider a natural model, namely the GARCH one, as it has been developed for modelling volatility clustering.

Given the aim of this study, we choose to use the simplest version, the GARCH(1,1), to model the log- returns $(X_t)_t$ as: $X_{t+1} = \epsilon_t \, \sigma_t$, where the variance σ_t^2 of X_t satisfies

$$\sigma_t^2 = \omega + \alpha \, X_t^2 + \beta \, \sigma_{t-1}^2, \quad \text{with} \quad \omega > 0, \, \alpha > 0, \, \beta > 0,$$

and where the error terms $(\epsilon_t)_t$ constitute a Gaussian white noise. Here again, one could use a more sophisticated model with Student-t error terms in order to better fit the tails, but mixing fat tails with volatility clustering would not help us isolate and identify the main cause of the procyclicality, hence this choice of standard Gaussian noise. To estimate our model, we fit the parameters ω, α, β to each full sample of the 11 indices, using a robust optimization method proposed by Zumbach in [20]. Compared to the other methods, we saw that the method by Zumbach was the most precise in reproducing the average annual realized volatility. As a side note, we remark that the robustness is especially important as other optimization methods may not always fulfill the stationarity condition for the GARCH process, namely $\alpha + \beta < 1$. From Table 8, we see that the optimization gives in all cases, parameters that fulfill the GARCH stationarity condition. The estimated parameters do not vary much from one index to the other (except for AUS, ITA and JPN). Further, we see that the annualized volatilities of the GARCH reproduce very well the historical ones on average, when estimated over 1000 replications of the GARCH process.

Table 8: Descriptive Statistics of the GARCH models for the 11 indices

	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA
GARCH Parameters											
ω [10^{-6}]	4.13	1.32	3.11	2.51	2.66	5.96	2.63	3.42	4.08	2.10	2.24
α [10^{-1}]	1.66	1.08	0.98	0.95	0.83	1.48	1.12	1.33	1.12	1.13	1.07
β [10^{-1}]	7.93	8.79	8.83	8.89	9.02	8.19	8.74	8.48	8.65	8.68	8.76
$\alpha + \beta$	*0.96*	*0.99*	*0.98*	*0.98*	*0.99*	*0.97*	*0.99*	*0.98*	*0.98*	*0.98*	*0.98*
Fitting Results											
Likelihood [10^4]	2.54	2.58	2.35	2.39	2.32	2.25	2.37	2.36	2.30	2.51	2.45
Volatility [%]	16.0	15.8	19.9	20.0	20.9	21.3	21.3	20.9	21.0	16.6	18.3
Historical [%]	16.0	15.9	20.0	20.1	21.1	21.3	21.6	21.0	21.1	16.7	18.5

Given those GARCH parameters, we replicate for each index 1000 simulated paths with the same sample size as the original data sample. On each of these replications, we apply the same statistical analyses as those done on the historical data in Section 4. This means we analyze the behavior of the SQP for varying parameters p and thresholds α, we look at the RMSE and the SQP-ratios; as these results are not central to our analysis, we do not present them here. Then we look closer at the correlation between annual realized volatility and the logarithm of the SQP-ratios; it is what is developed next. All in all, we observe qualitatively similar results with the GARCH simulations as with the historical data.

5.1 Dependence between SQP Ratios and Annualized Volatility in the GARCH Model

As in §4.2, we first look at the Pearson correlation between the annualized daily volatility and the logarithm of the SQP ratios computed on a 1-year sample. Those are depicted in Table 9. As for historical data, we observe a negative correlation here too. The negative Pearson correlations (again, using the logarithm of the SQP ratios) of the GARCH for the averages are higher (in absolute value) than in the data (see Table 6), the difference between the two increasing with p. Furthermore, as in the case of the historical data, the negative correlation is stronger for the lower powers p - except that, in contrast to the historical data, the behavior for $\alpha = 0.95$ is different for $p = 1, 2$ (while it was strongly decreasing with p for historical data, here it remains at similar values).

Table 9: Average Pearson correlation between the annualized volatility and the log of SQP ratios computed on a 1-year sample, using a GARCH(1,1) model, and for two thresholds (95 and 99%)

AVG$\pm\sigma$	$\alpha = 0.95$	$\alpha = 0.99$
$p = 0$	**-0.63 $\pm$ 0.07**	**-0.61 $\pm$ 0.07**
$p = 0.5$	**-0.66 $\pm$ 0.07**	**-0.61 $\pm$ 0.07**
$p = 1$	**-0.66 $\pm$ 0.07**	**-0.60 $\pm$ 0.07**
$p = 2$	**-0.63 $\pm$ 0.07**	**-0.60 $\pm$ 0.07**

As before, we also illustrate the obtained results with the S&P 500 index, considering one (simulated) sample path of the fitted GARCH(1,1) model, and look at the various realized SQP ratios (as well as its logarithm) for $p = 0$, as a function of the annual realized volatility; see Figure 6. For a better comparison between the GARCH realization and the historical data, we use the same y-scale for both. The experiment has been replicated with many sample plaths and gives the same conclusion.

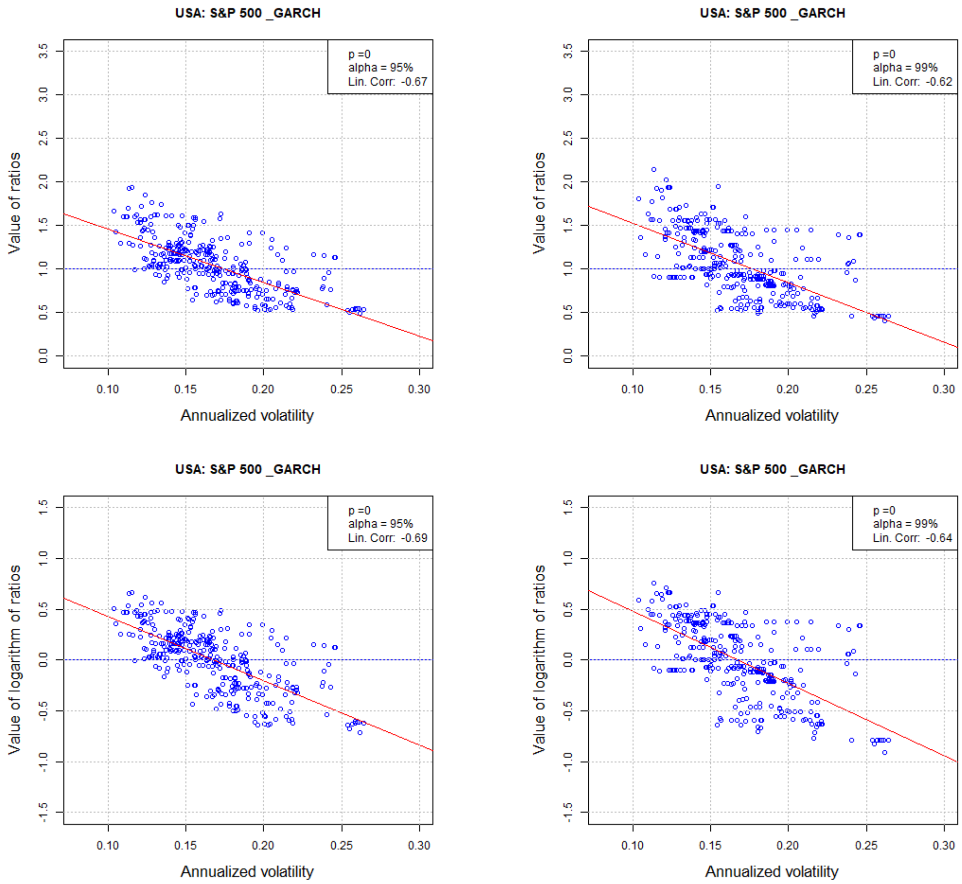

Figure 6: SQP Ratios as a function of annualized volatility for one simulated sample path of the GARCH(1,1) (that fits the S&P500) for $p = 0$: On the left $\alpha = 95\%$ and on the right $\alpha = 99\%$; first row SQP-ratios, second row logarithm of SQP-ratios

We observe a similar behavior for the GARCH realization (Fig.6) and the historical data (Fig.5):

- We have a negative correlation in both cases and the slope of the simple linear regression lines look similar.
- The more volatility there is in year t, the lower are the ratios $R_{p,\alpha,T,t}$, which means that losses in year $t+1$ have been overestimated with the measures calculated in year t.
- This overestimation can result in ratios below 0.5, *i.e.* in the above realization the risk computed at the height of the crisis is more than double the size of the risk measured a year later.
- For the underestimation, we see that the realized ratios can be larger than 2; in other

words, the risk next year is more than 2 times the risk measured during the current year for this sample.

As can be seen from Figure 6, the linear correlation is not an optimal approximation of the dependence structure (as already observed for historical data, although slightly stronger for GARCH). Thus, we consider the log SQP ratio to take into account the non linear dependence and, additionally, compute a rank correlation - see e.g. Table 10.

Table 10: Average Spearman's ρ rank correlation between the annualized volatility and the SQP ratios computed on a 1-year sample, using a GARCH(1,1) model

AVG$\pm\sigma$	$\alpha = 0.95$	$\alpha = 0.99$
$p = 0$	**-0.61 $\pm$ 0.08**	**-0.59 $\pm$ 0.08**
$p = 0.5$	**-0.64 $\pm$ 0.08**	**-0.59 $\pm$ 0.08**
$p = 1$	**-0.64 $\pm$ 0.08**	**-0.58 $\pm$ 0.09**
$p = 2$	**-0.61 $\pm$ 0.09**	**-0.58 $\pm$ 0.09**

For the GARCH(1,1) model, the rank correlation is of the same order (slightly lower) than the Pearson correlation. This is similar to the historical data (there the differences were a bit more pronounced, especially for low α and low p). On average (i.e. when looking at the average values of the 11 indices considered) we see the same behavior with respect to α and p for the Spearman as well as the Pearson correlation in this case.

Overall, a simple GARCH model captures pretty well the main features we have seen in the data. Thus, this procyclicality seems related to the clustering of volatility and its return to the mean. Despite having the same tendencies, we still have seen differences between the two. Thus, up to now, we do not have evidence to claim that the procyclicality in the data is solely due to the GARCH effect.

6 Conclusion

Popularized by JP Morgan in 1996 (see [19]), the Value-at-Risk measure is still, nowadays, mostly used by banks and some insurance companies for risk management purposes. This use has been reinforced by regulators even if it has been shown that Value-at-Risk was not a coherent measure contrary to Expected Shortfall in [2]. However, as a lesson learned from the 2008 financial crisis, the Basel Committee is moving towards Expected Shortfall. Nonetheless, this study remains timely since VaR is the measure used by both banks and insurances and our analysis could be generalized for Expected Shortfall at a later stage. However, the question remains if measuring risk using a risk measure on historical data is appropriate to forecast the risk financial institutions face. In this study, we show empirically that risk measures based on VaR are in fact pro-cyclical. Looking at 11 stock indices, we find that in times of low volatility, the risk is underestimated, while in times of crises the risk is overestimated.

Although pro-cyclicality of risk models has widely been assumed by market actors, it was never clearly quantified, nor shown empirically with enough evidence. Our way to quantify this effect is to model the risk measure as a stochastic process and condition it on realized volatility. We use the Sample Quantile Process (SQP), a 'dynamic generalization of VaR', that can be defined in many ways depending on the choice of measure μ that is part of its definition. After studying various choices of parameters, we conduct a study of the dynamics of risk measures as a function of realized volatility. To measure the so called predictive power, we introduce a new quantity, a ratio of risk measures to quantify the quality of risk prediction. While we only used it for the SQP/VaR, it is a universal quantity that can be applied to any other risk measure like the Expected Shortfall without restrictions. It is a novel way of measuring the accuracy of risk estimation that does not rely on number of violations. Our study confirms the assumption presented above and allows us to quantify this effect.

We observed that simple GARCH(1,1) models provide similar characteristics as in the data, i.e. we saw a negative correlation between realized volatility and the SQP-ratios with even slightly higher values than in the historical data. We thus relate this effect to the volatility clustering present in financial markets. Still, we do not have evidence to claim that the procyclicality in the data is solely due to the GARCH effect - further and more statistical investigations are in progress. As our observations are confirmed on 11 different stock indices we can conclude that the risk management imposed by regulators favors pro-cyclical behaviors of the market actors. Indeed, it does not reflect the genuine potential risks and losses. Following a pure historical estimation of risk measures, banks would be required to keep more capital than needed during a crisis (period of high volatility), while the requirements during quiet period grossly underestimate the risk, leaving the banks, or financial institutions in general, unprepared in case of crisis.

Finally, having empirically quantified a pro-cyclical effect of the sample VaR, we can now take it into account to suggest an anti-cyclical risk measure. To conclude, in view of our results, the introduction of anti-cyclical risk management measures, which are at the opposite of what is currently required for financial institutions by regulators, would better prepare the financial system to cope with future crises by enhancing the capital requirements in quiet times and relaxing them during crises. Clearly, these measures should be in line with the empirical results without hampering the system or introducing rules that weaken economic valuation of liabilities as it is currently done. This is what we are developing. Studies are under way to design SQP with the right dynamical behavior that would be a good basis for anti-cyclical regulation.

Acknowledgement: We are grateful to Prof. Jiro Akahori for his suggestion of using the SQP and for fruitful discussions on it.

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Appendix A: Tables

A.1 Rank Correlation on Real Data

Table A.1: Spearman rank correlation between the SQP ratios and the volatility, for each index and for $T = 1$ year, over the whole historical sample, and for two thresholds (95 and 99%). In the last column, we present the average over all indices ± the standard deviation.

	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG (± sd)**
						$\alpha = 95\%$						
p=0	-0.44	-0.40	-0.47	-0.43	-0.47	-0.60	-0.48	-0.51	-0.53	-0.44	-0.41	**-0.47 ± 0.06**
p=0.5	-0.47	-0.39	-0.45	-0.42	-0.39	-0.63	-0.50	-0.50	-0.50	-0.47	-0.36	**-0.46 ± 0.07**
p=1	-0.51	-0.38	-0.36	-0.30	-0.27	-0.53	-0.47	-0.56	-0.39	-0.46	-0.38	**-0.42 ± 0.09**
p=2	-0.56	-0.36	-0.35	-0.25	-0.22	-0.47	-0.34	-0.51	-0.28	-0.37	-0.41	**-0.37 ± 0.11**
						$\alpha = 99\%$						
p=0	-0.50	-0.46	-0.48	-0.47	-0.53	-0.58	-0.51	-0.52	-0.55	-0.44	-0.47	**-0.50 ± 0.04**
p=0.5	-0.55	-0.43	-0.44	-0.43	-0.36	-0.49	-0.40	-0.51	-0.44	-0.39	-0.49	**-0.45 ± 0.06**
p=1	-0.54	-0.47	-0.45	-0.38	-0.38	-0.52	-0.40	-0.51	-0.47	-0.37	-0.51	**-0.46 ± 0.06**
p=2	-0.54	-0.47	-0.45	-0.38	-0.38	-0.52	-0.40	-0.51	-0.47	-0.37	-0.51	**-0.46 ± 0.06**

Table A.2: Kendall rank correlation between the SQP ratios and the volatility, for each index and for $T = 1$ year, over the whole historical sample, and for two thresholds (95 and 99%). In the last column, we present the average over all indices ± the standard deviation.

	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG (± sd)**
						$\alpha = 95\%$						
p=0	-0.32	-0.27	-0.33	-0.30	-0.33	-0.45	-0.33	-0.35	-0.37	-0.31	-0.29	**-0.33 ± 0.05**
p=0.5	-0.34	-0.26	-0.31	-0.28	-0.27	-0.47	-0.35	-0.35	-0.34	-0.34	-0.26	**-0.33 ± 0.06**
p=1	-0.37	-0.27	-0.24	-0.21	-0.20	-0.39	-0.33	-0.40	-0.27	-0.33	-0.26	**-0.30 ± 0.07**
p=2	-0.39	-0.25	-0.24	-0.17	-0.16	-0.34	-0.24	-0.35	-0.18	-0.26	-0.27	**-0.26 ± 0.07**
						$\alpha = 99\%$						
p=0	-0.37	-0.32	-0.34	-0.32	-0.36	-0.44	-0.36	-0.37	-0.39	-0.32	-0.34	**-0.36 ± 0.04**
p=0.5	-0.39	-0.31	-0.31	-0.29	-0.23	-0.35	-0.28	-0.36	-0.29	-0.28	-0.32	**-0.31 ± 0.04**
p=1	-0.39	-0.34	-0.32	-0.26	-0.25	-0.38	-0.27	-0.36	-0.32	-0.26	-0.34	**-0.32 ± 0.05**
p=2	-0.39	-0.34	-0.32	-0.26	-0.25	-0.38	-0.27	-0.36	-0.32	-0.26	-0.34	**-0.32 ± 0.05**

A.2 Rank and Linear Correlation on Simulated GARCH

Table A.3: Pearson correlation between the logarithm of the SQP ratios and the volatility, for each index and for $T = 1$ year, using a GARCH(1,1) model (average over 1000 realizations), and for two thresholds (95 and 99%). In the last column, we present the average over all indices $\pm$ the standard deviation.

Mean for	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG ($\pm$ sd)**
						$\alpha = 95\%$						
p=0	-0.64	-0.62	-0.63	-0.63	-0.62	-0.64	-0.63	-0.63	-0.64	-0.63	-0.63	**-0.63 $\pm$ 0.07**
p=0.5	-0.69	-0.65	-0.65	-0.65	-0.64	-0.68	-0.65	-0.67	-0.67	-0.66	-0.66	**-0.66 $\pm$ 0.07**
p=1	-0.70	-0.65	-0.65	-0.65	-0.64	-0.69	-0.66	-0.68	-0.67	-0.66	-0.66	**-0.66 $\pm$ 0.08**
p=2	-0.66	-0.62	-0.61	-0.62	-0.60	-0.65	-0.63	-0.64	-0.63	-0.63	-0.63	**-0.63 $\pm$ 0.09**
						$\alpha = 99\%$						
p=0	-0.62	-0.60	-0.60	-0.60	-0.60	-0.62	-0.61	-0.61	-0.61	-0.61	-0.61	**-0.61 $\pm$ 0.07**
p=0.5	-0.63	-0.60	-0.59	-0.60	-0.59	-0.62	-0.61	-0.61	-0.61	-0.61	-0.61	**-0.61 $\pm$ 0.08**
p=1	-0.62	-0.59	-0.59	-0.59	-0.58	-0.61	-0.60	-0.61	-0.60	-0.60	-0.60	**-0.60 $\pm$ 0.08**
p=2	-0.62	-0.59	-0.59	-0.59	-0.58	-0.61	-0.60	-0.61	-0.60	-0.60	-0.60	**-0.60 $\pm$ 0.08**

Table A.4: Spearman rank correlation between the SQP ratios and the volatility, for each index and for $T = 1$ year, using a GARCH(1,1) model (average over 1000 realizations), and for two thresholds (95 and 99%). In the last column, we present the average over all indices $\pm$ the standard deviation.

Mean for	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG ($\pm$ sd)**
						$\alpha = 95\%$						
p=0	-0.62	-0.61	-0.61	-0.61	-0.60	-0.61	-0.61	-0.62	-0.62	-0.61	-0.61	**-0.61 $\pm$ 0.08**
p=0.5	-0.66	-0.63	-0.63	-0.63	-0.62	-0.65	-0.63	-0.65	-0.64	-0.64	-0.64	**-0.64 $\pm$ 0.08**
p=1	-0.66	-0.63	-0.63	-0.63	-0.62	-0.65	-0.64	-0.65	-0.64	-0.64	-0.64	**-0.64 $\pm$ 0.08**
p=2	-0.63	-0.61	-0.59	-0.60	-0.59	-0.62	-0.61	-0.62	-0.61	-0.61	-0.61	**-0.61 $\pm$ 0.09**
						$\alpha = 99\%$						
p=0	-0.59	-0.59	-0.58	-0.59	-0.58	-0.59	-0.59	-0.60	-0.59	-0.59	-0.59	**-0.59 $\pm$ 0.08**
p=0.5	-0.59	-0.58	-0.57	-0.58	-0.57	-0.59	-0.59	-0.59	-0.59	-0.59	-0.59	**-0.59 $\pm$ 0.08**
p=1	-0.59	-0.58	-0.57	-0.57	-0.56	-0.58	-0.58	-0.59	-0.58	-0.58	-0.58	**-0.58 $\pm$ 0.09**
p=2	-0.59	-0.58	-0.57	-0.57	-0.56	-0.58	-0.58	-0.59	-0.58	-0.58	-0.58	**-0.58 $\pm$ 0.09**

Table A.5: Kendall rank correlation between the SQP ratios and the volatility, for each index and for $T = 1$ year, using a GARCH(1,1) model (average over 1000 realizations), and for two thresholds (95 and 99%). In the last column, we present the average over all indices $\pm$ the standard deviation.

Mean for	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG ($\pm$ sd)**
						$\alpha = 95\%$						
p=0	-0.45	-0.44	-0.44	-0.44	-0.44	-0.45	-0.44	-0.45	-0.45	-0.45	-0.45	**-0.44 $\pm$ 0.06**
p=0.5	-0.48	-0.46	-0.45	-0.46	-0.45	-0.47	-0.46	-0.47	-0.47	-0.46	-0.46	**-0.46 $\pm$ 0.06**
p=1	-0.49	-0.46	-0.45	-0.46	-0.45	-0.48	-0.46	-0.48	-0.47	-0.46	-0.46	**-0.46 $\pm$ 0.07**
p=2	-0.45	-0.44	-0.42	-0.43	-0.42	-0.44	-0.44	-0.45	-0.44	-0.44	-0.44	**-0.44 $\pm$ 0.07**
						$\alpha = 99\%$						
p=0	-0.43	-0.42	-0.42	-0.42	-0.42	-0.43	-0.42	-0.43	-0.43	-0.43	-0.42	**-0.42 $\pm$ 0.06**
p=0.5	-0.43	-0.42	-0.41	-0.42	-0.41	-0.42	-0.42	-0.43	-0.42	-0.42	-0.42	**-0.42 $\pm$ 0.07**
p=1	-0.42	-0.41	-0.40	-0.41	-0.40	-0.42	-0.42	-0.42	-0.41	-0.41	-0.42	**-0.41 $\pm$ 0.07**
p=2	-0.42	-0.41	-0.40	-0.41	-0.40	-0.42	-0.42	-0.42	-0.41	-0.41	-0.42	**-0.41 $\pm$ 0.07**

Appendix B: Figures (1 year sample)

B.1 SQP on a Rolling Window

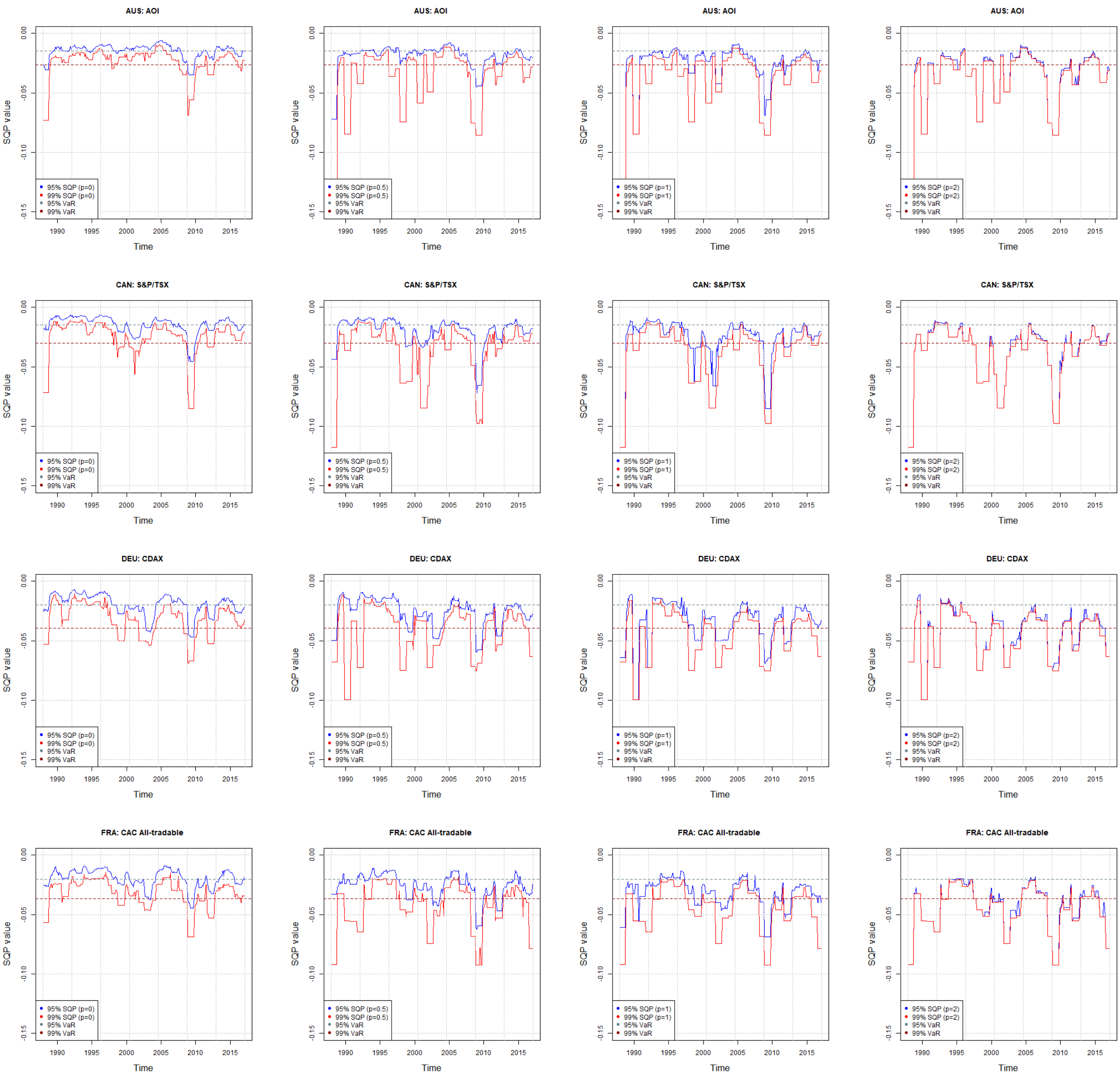

Figure A.1: SQP rolling every month with $T = 1$ year, and thresholds $\alpha = 95\%$ and 99% for different indices and values of p. From left to right: $p = 0$, $p = 0.5$, $p = 1$ and $p = 2$. From top to bottom: AUS, CAN, DEU, FRA. Dashed lines correspond to VaR(α) computed over the whole sample.

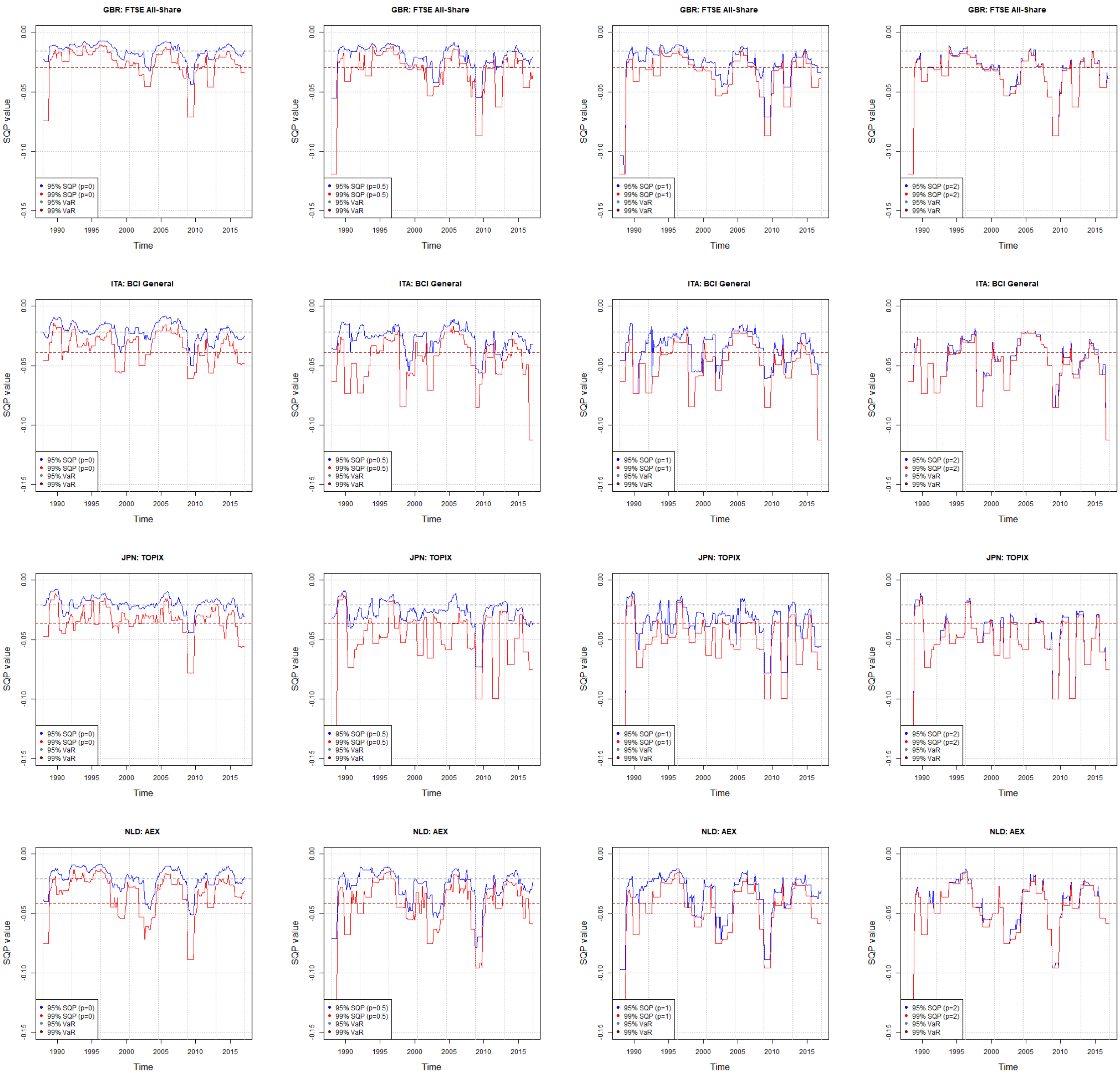

Figure A.2: SQP rolling every month with $T = 1$ year, and thresholds $\alpha = 95\%$ and 99% for different indices and values of p. From left to right: $p = 0$, $p = 0.5$, $p = 1$ and $p = 2$. From top to bottom: GBR, ITA, JPN, NLD. Dashed lines correspond to VaR(α) computed over the whole sample.

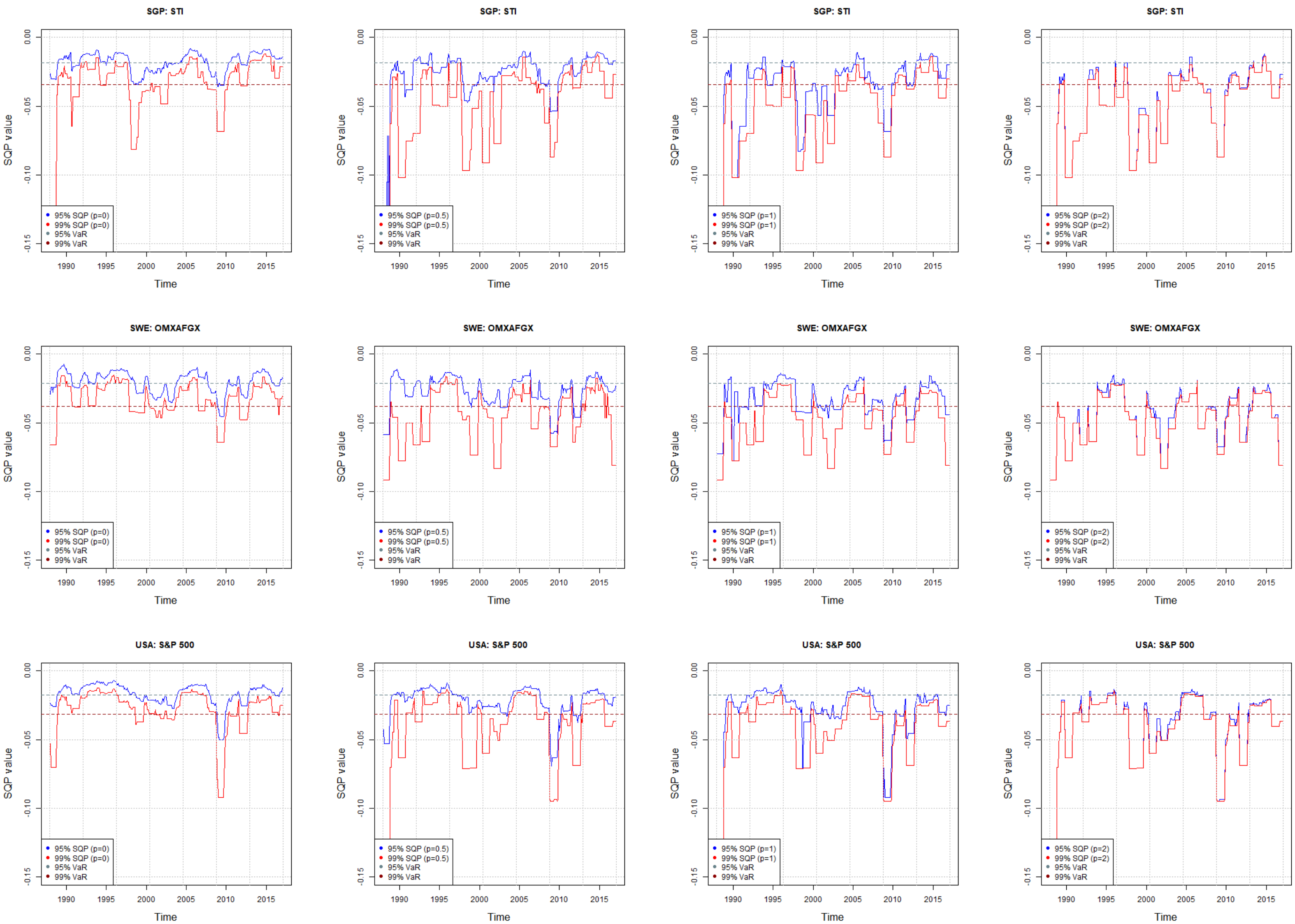

Figure A.3: SQP rolling every month with $T = 1$ year, and thresholds $\alpha = 95\%$ and 99% for different indices and values of p. From left to right: $p = 0$, $p = 0.5$, $p = 1$ and $p = 2$. From top to bottom: SGP, SWE, USA. Dashed lines correspond to VaR(α) computed over the whole sample.

B.2 Mean Value of the SQP

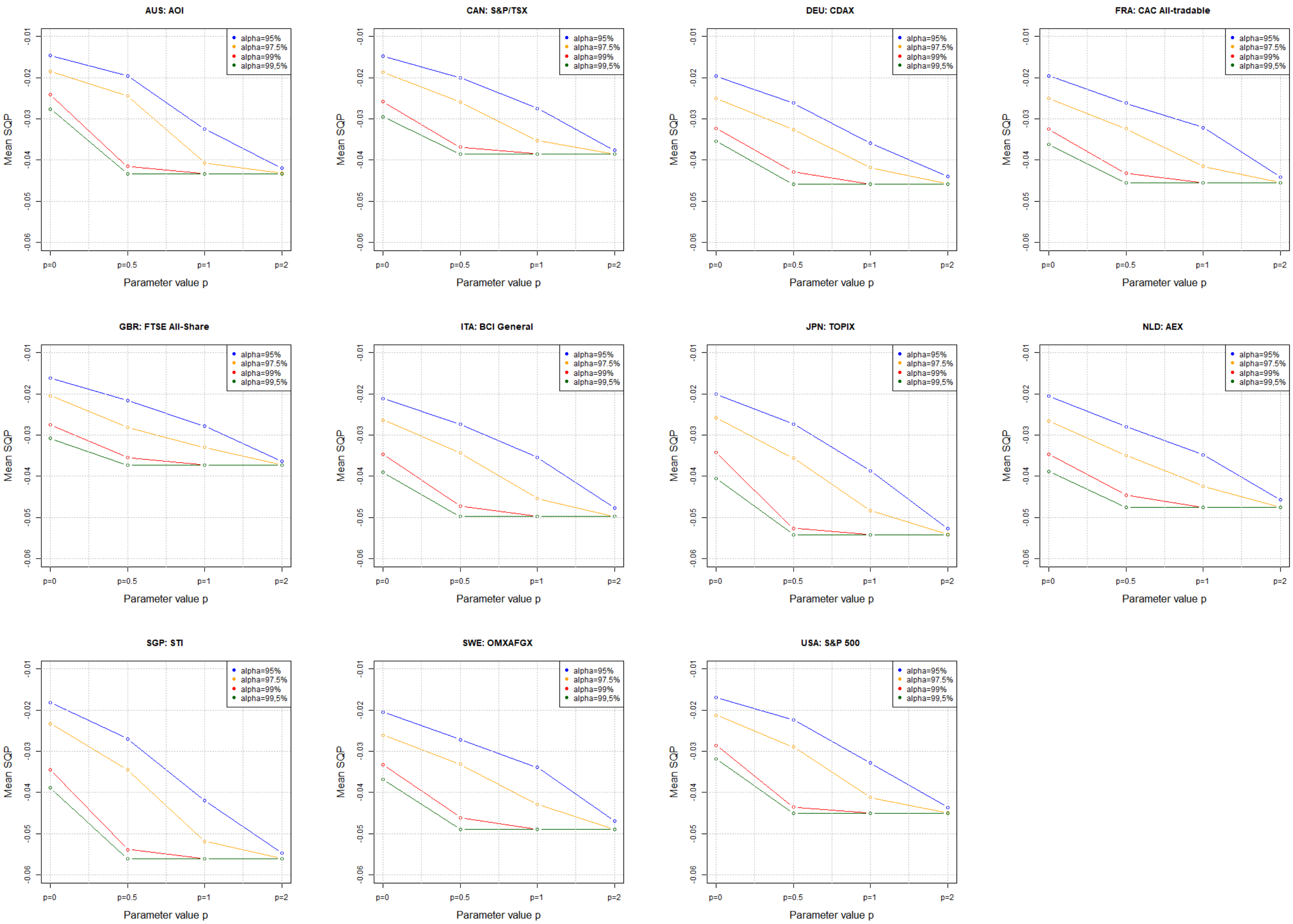

Figure A.4: Average over the whole sample of rolling-window SQPs with sample size $T = 1y$ as a function of the power p for various thresholds α. Each figure is for one index, shown in alphabetical order: AUS, CAN, DEU, FRA, GBR, ITA, JPN, NLD, SGP, SWE, USA.

B.3 RMSE of SQP Ratios

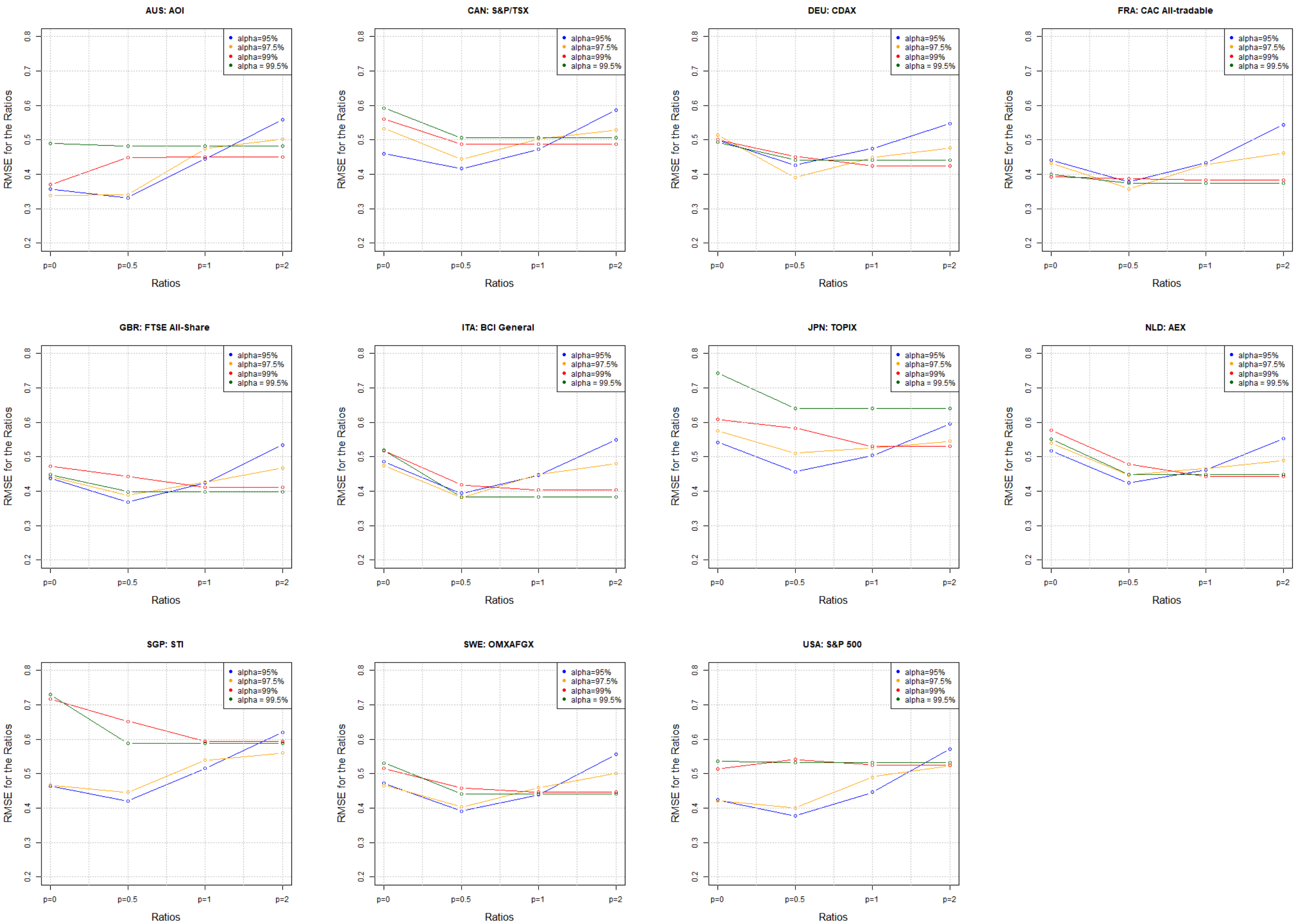

Figure A.5: RMSE (defined in (3.2)) of the SQP Ratios $R_{p,\alpha,1}(t)$ taking $p = 0, 0.5, 1, 2$ and two thresholds $\alpha = 95\%$ (in blue), 99% (in red). Each figure is for one index, shown in alphabetical order: AUS, CAN, DEU, FRA, GBR, ITA, JPN, NLD, SGP, SWE, USA.

B.4 Annualized Volatility

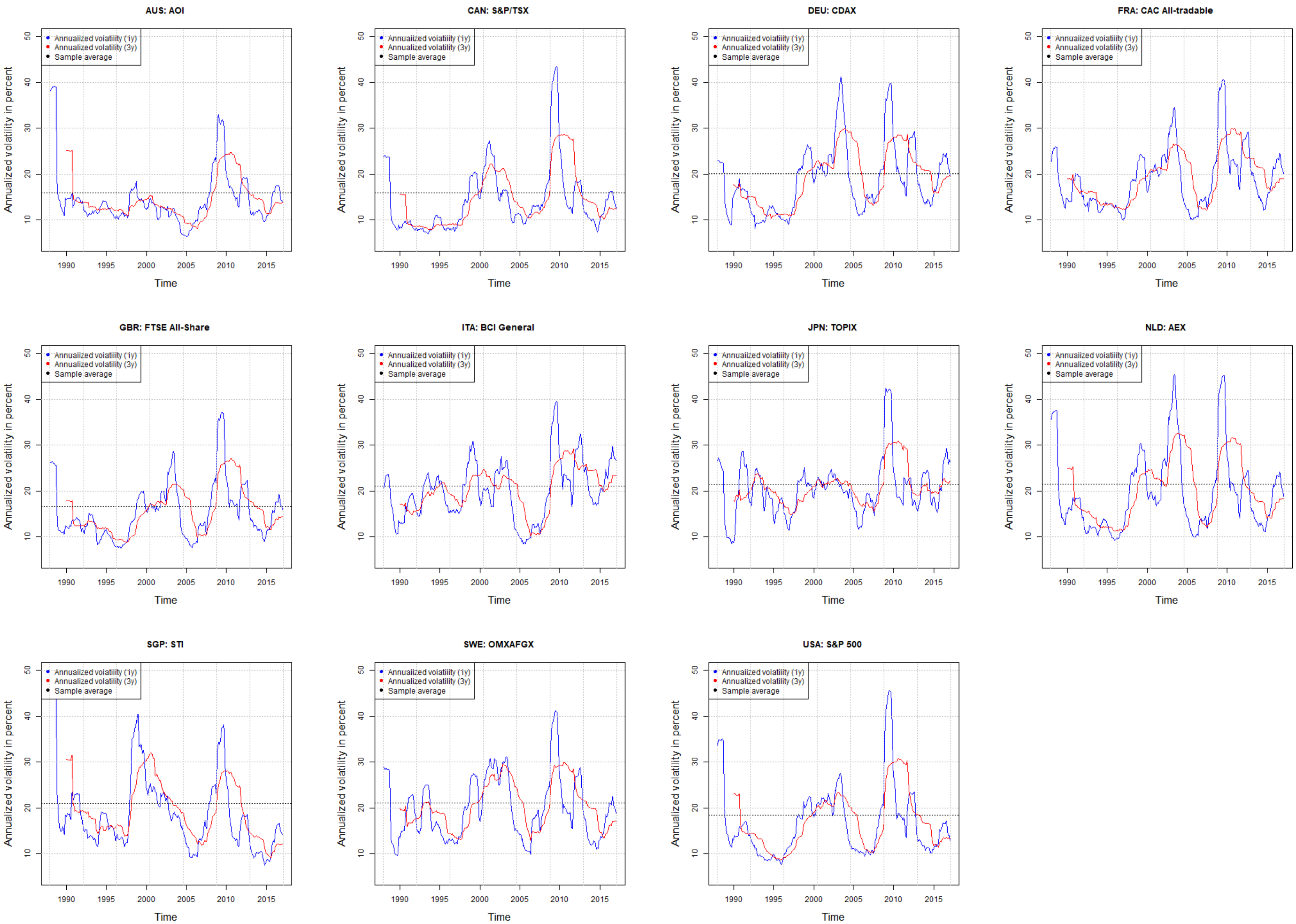

Figure A.6: Annual realized volatility (defined in (4.1)) over a rolling sample of size $T = 1, 3$ year(s). The dashed line represents the sample average of the volatility. Each figure is for one index, shown in alphabetical order: AUS, CAN, DEU, FRA, GBR, ITA, JPN, NLD, SGP, SWE, USA.

B.5 SQP Ratios and Annualized Volatility

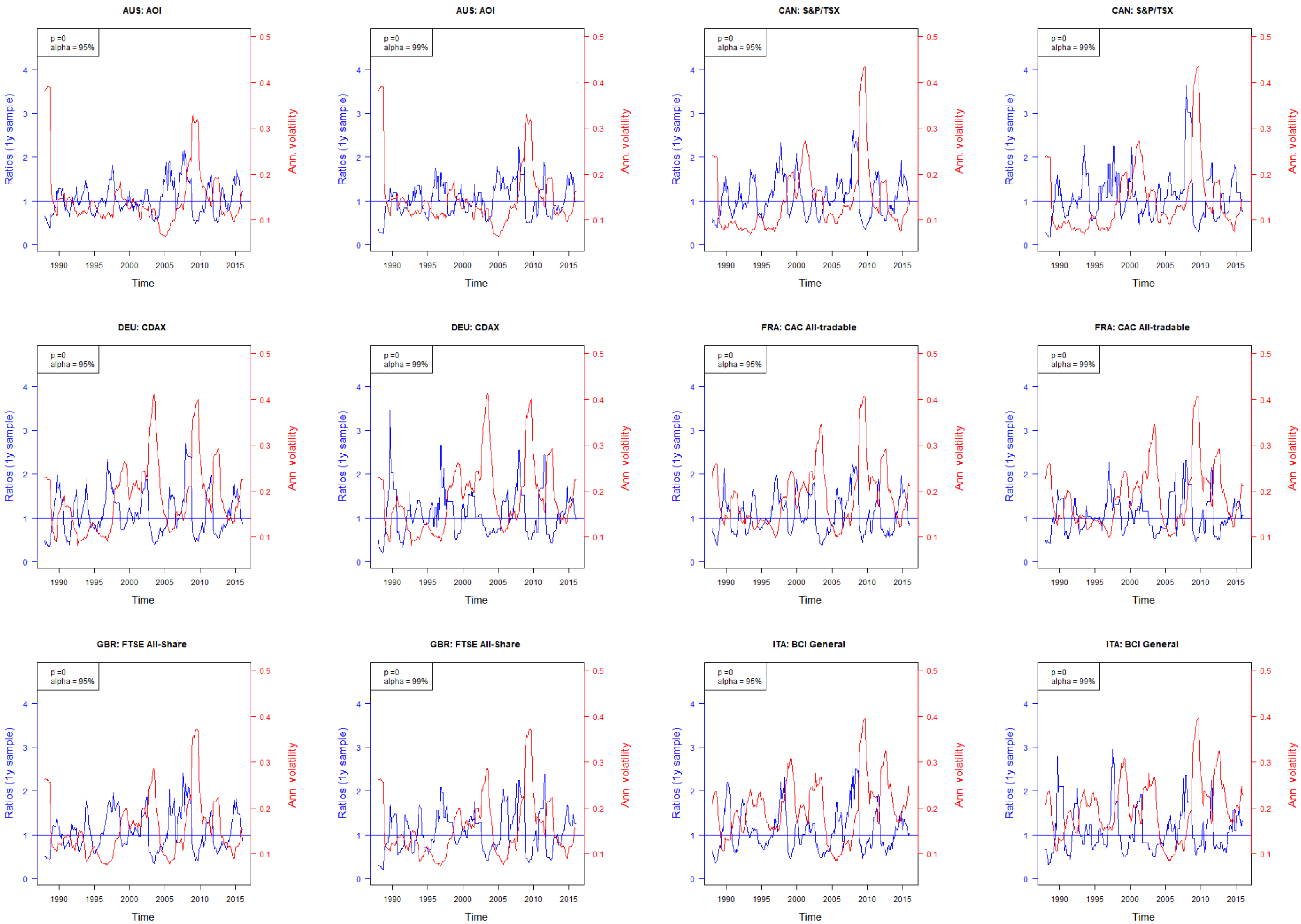

Figure A.7: SQP Ratios and annualized volatility time series computed over 1 year ($T = 1y$) for $\alpha = 95\%$ and $\alpha = 99\%$. The two figures corresponding to an index are shown next to each other, i.e. we have two indices per row. The indices are shown in alphabetical order: AUS, CAN, DEU, FRA,GBR, ITA.

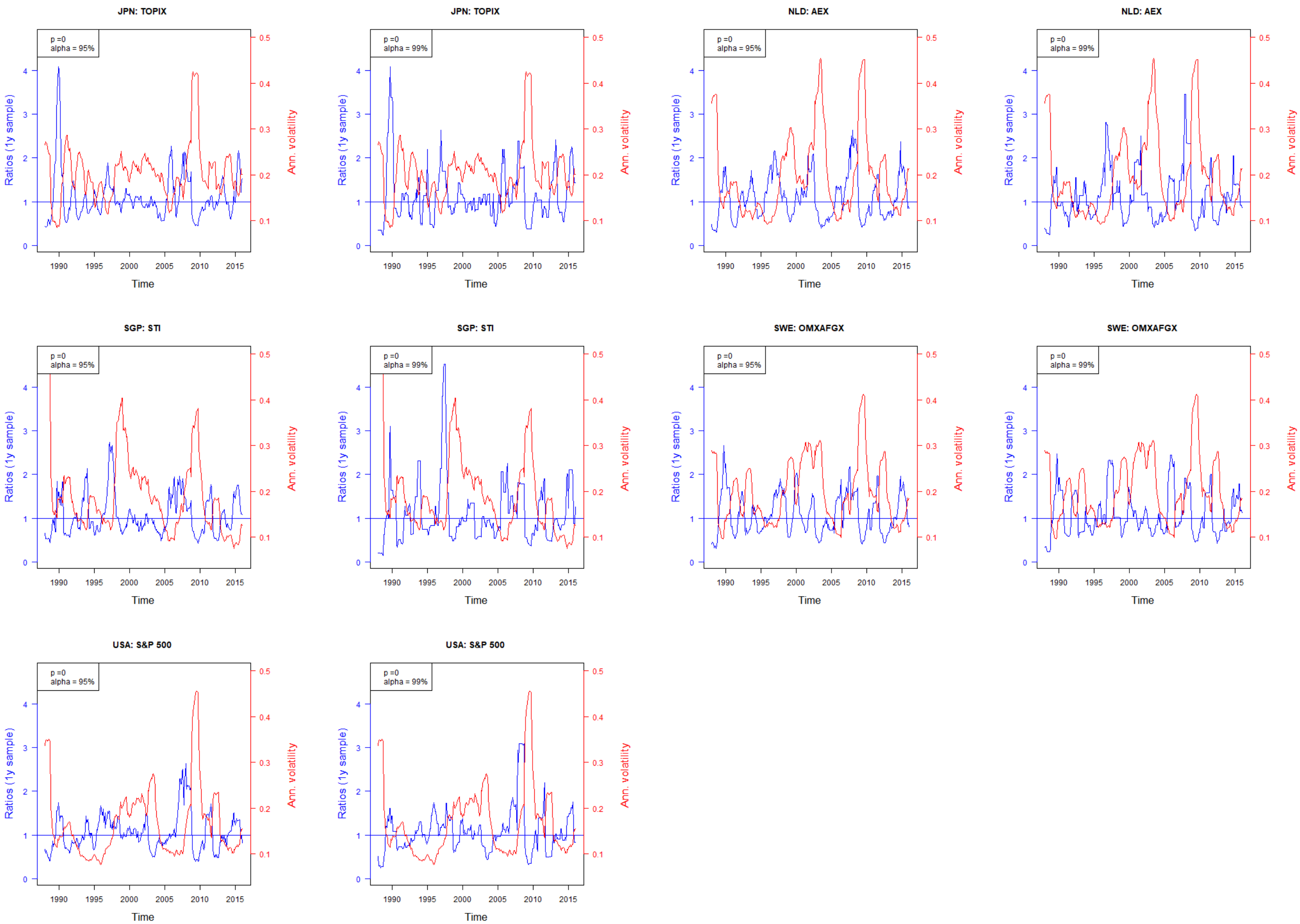

Figure A.8: SQP Ratios and annualized volatility time series computed over 1 year ($T = 1y$) for $\alpha = 95\%$ and $\alpha = 99\%$. The two figures corresponding to an index are shown next to each other, i.e. we have two indices per row. The indices are shown in alphabetical order: JPN, NLD, SGP, SWE, USA

B.6 SQP Ratio vs. Annualized Volatility

B.6.1 On real data

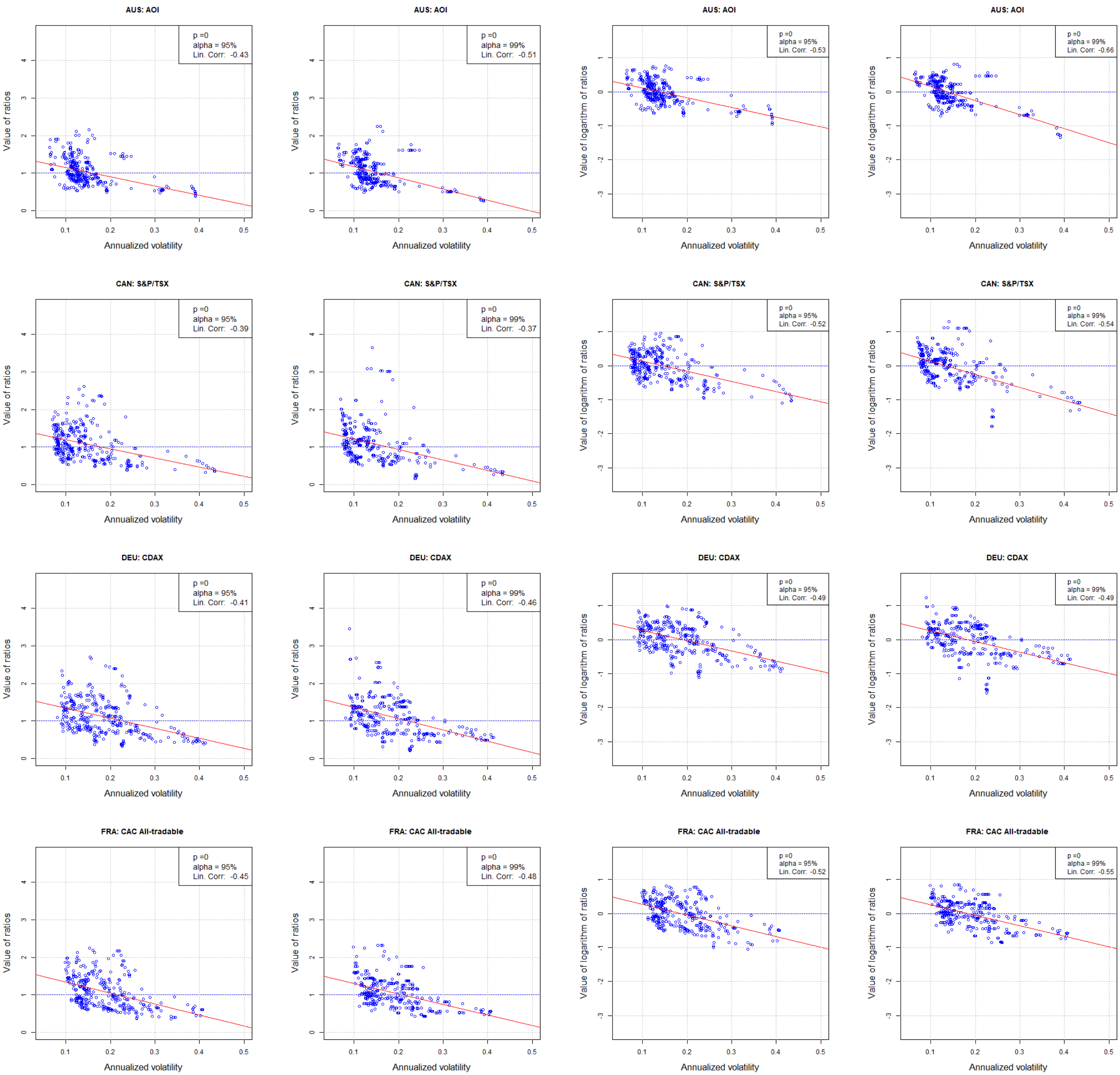

Figure A.9: SQP Ratios as a function of annualized volatility for $p = 0$ for $\alpha = 95\%, 99\%$. In each row we consider the SQP Ratios and then the logarithm of the SQP Ratios: From left to right, SQP ratio for $\alpha = 95\%, 99\%$, then logarithm of SQP ratio for $\alpha = 95\%, 99\%$. From top to bottom: AUS, CAN, DEU, FRA.

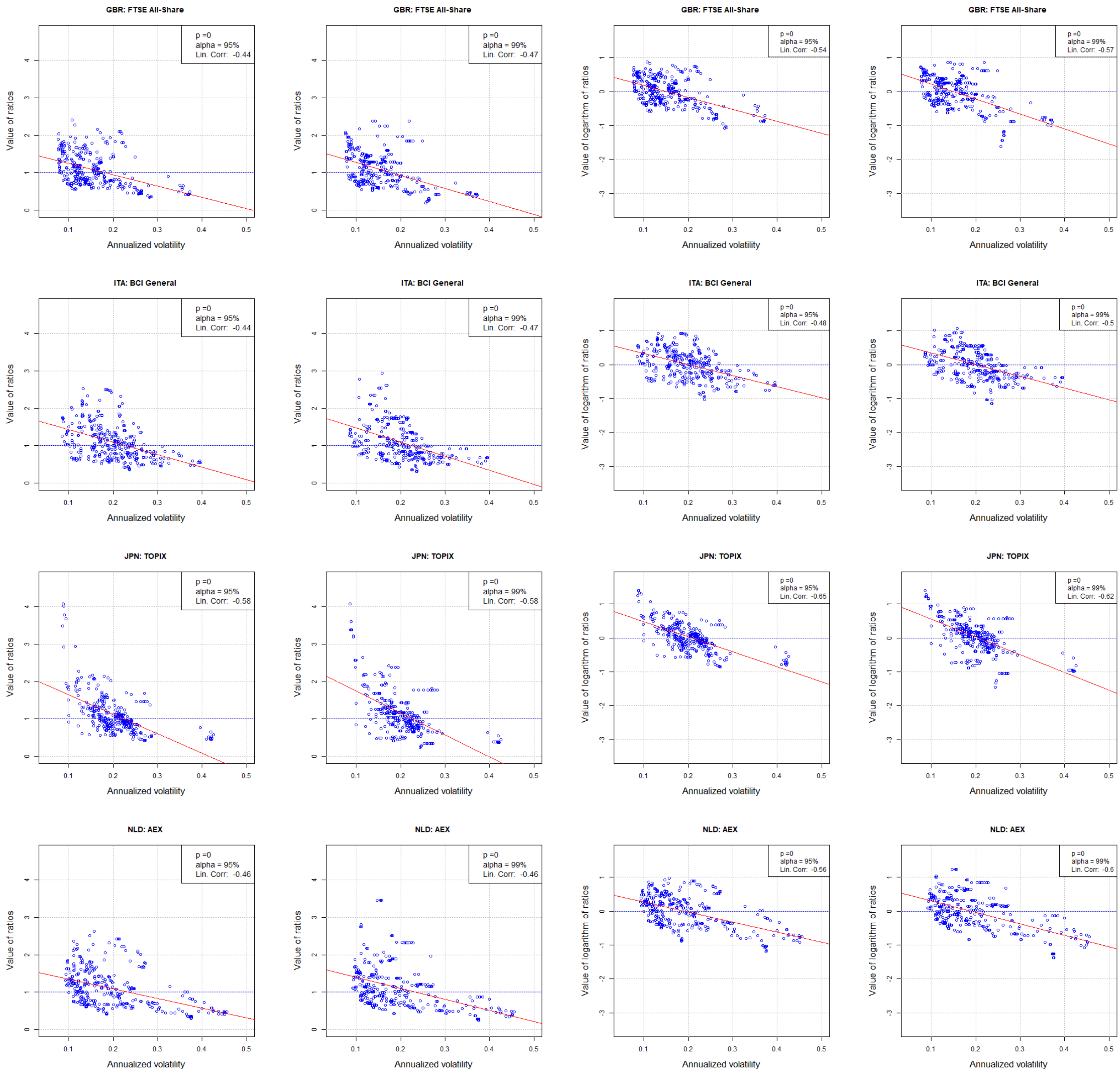

Figure A.10: SQP Ratios as a function of annualized volatility for $p = 0$ for $\alpha = 95\%, 99\%$. In each row we consider the SQP Ratios and then the logarithm of the SQP Ratios: From left to right, SQP ratio for $\alpha = 95\%, 99\%$, then logarithm of SQP ratio for $\alpha = 95\%, 99\%$. From top to bottom: GBR, ITA, JPN, NLD.

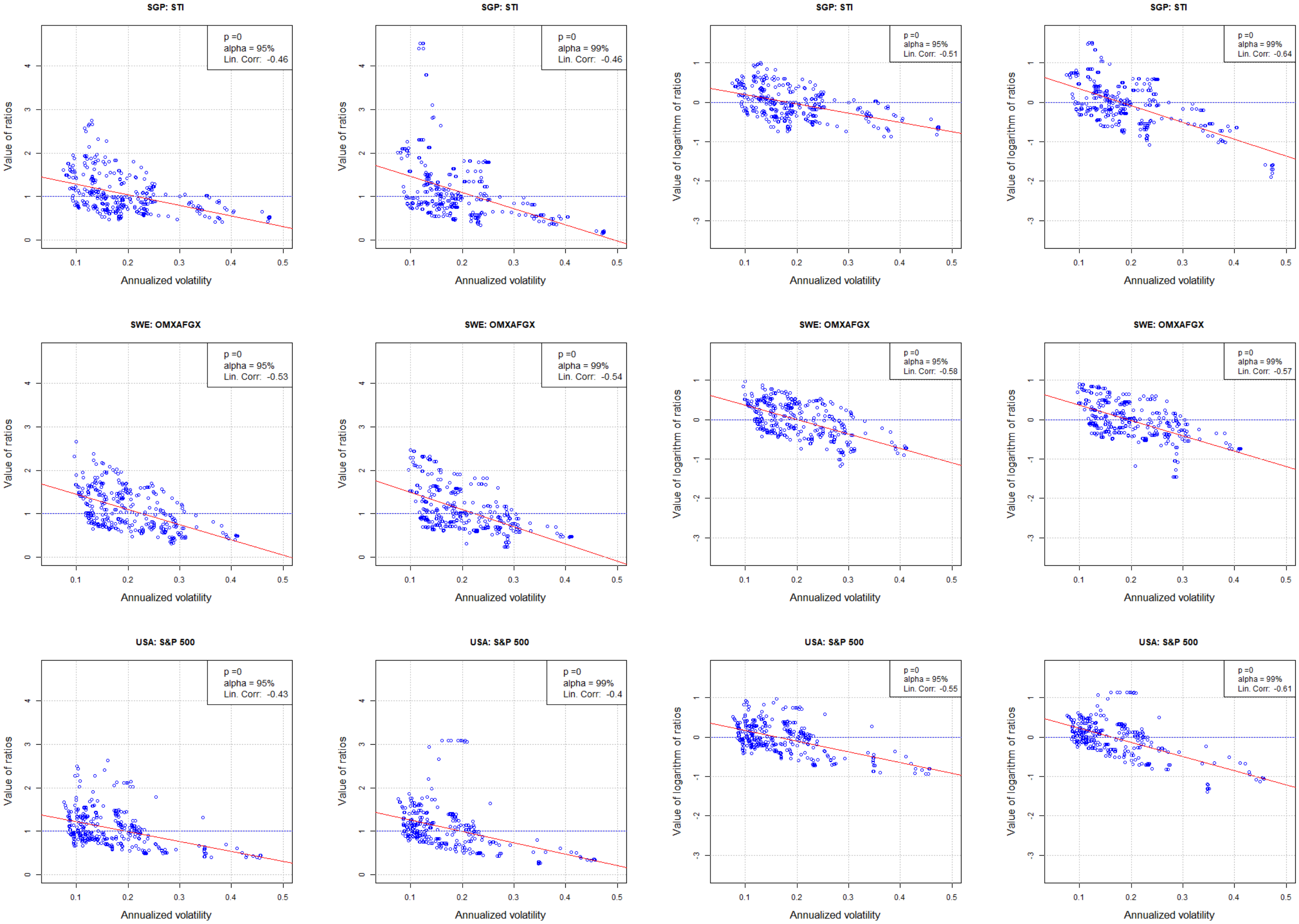

Figure A.11: SQP Ratios as a function of annualized volatility for $p = 0$ for $\alpha = 95\%, 99\%$. In each row we consider the SQP Ratios and then the logarithm of the SQP Ratios: From left to right, SQP ratio for $\alpha = 95\%, 99\%$, then logarithm of SQP ratio for $\alpha = 95\%, 99\%$. From top to bottom: SGP, SWE, USA

B.6.2 On a simulated GARCH realization

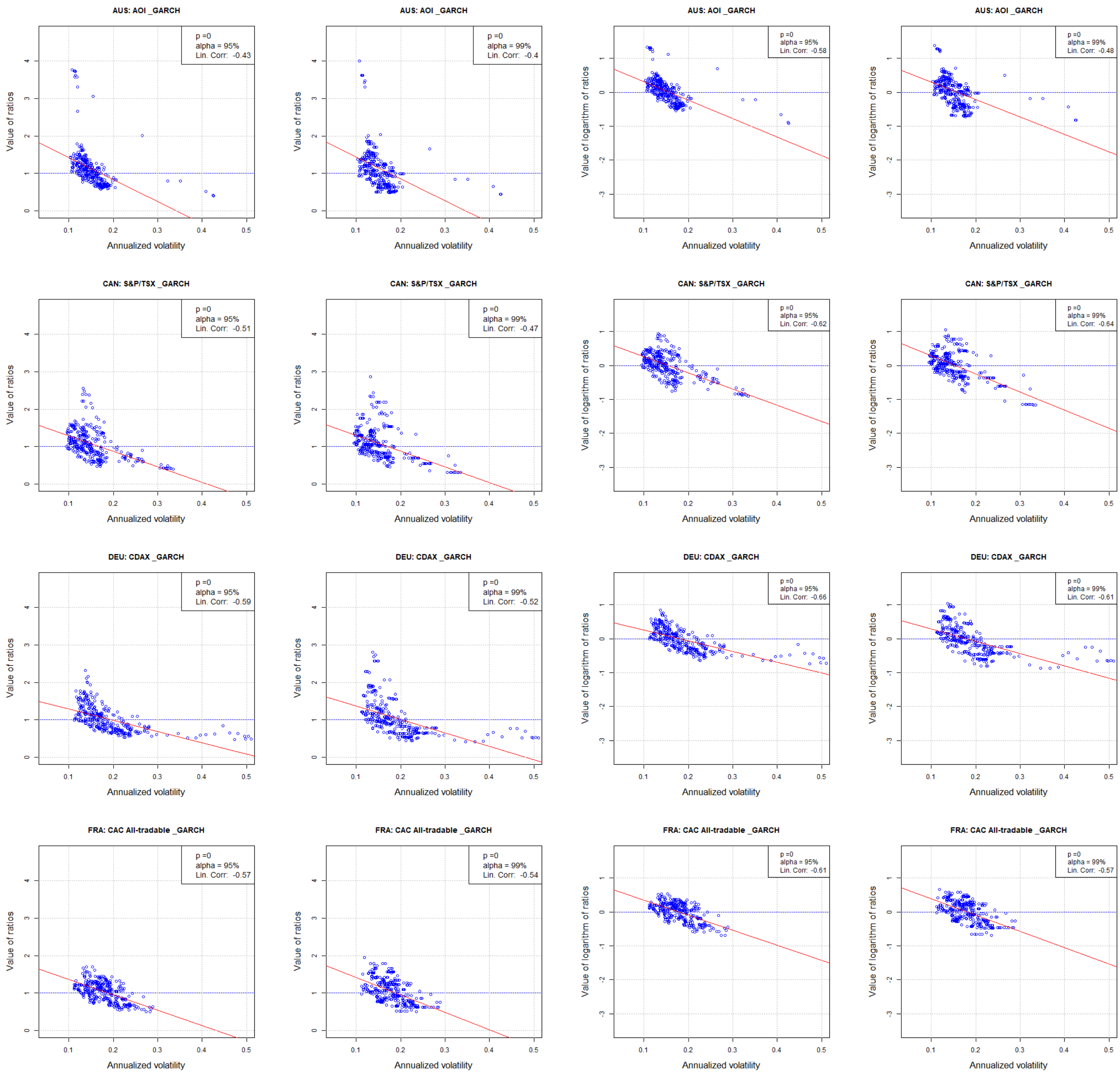

Figure A.12: SQP Ratios as a function of annualized volatility for one simulated sample path of the GARCH(1,1) - which is fitted to each index separetely - for $p = 0$ for $\alpha = 95\%, 99\%$. In each row we consider the SQP Ratios and then the logarithm of the SQP Ratios: From left to right, SQP ratio for $\alpha = 95\%, 99\%$, then logarithm of SQP ratio for $\alpha = 95\%, 99\%$. From top to bottom: AUS, CAN, DEU, FRA.

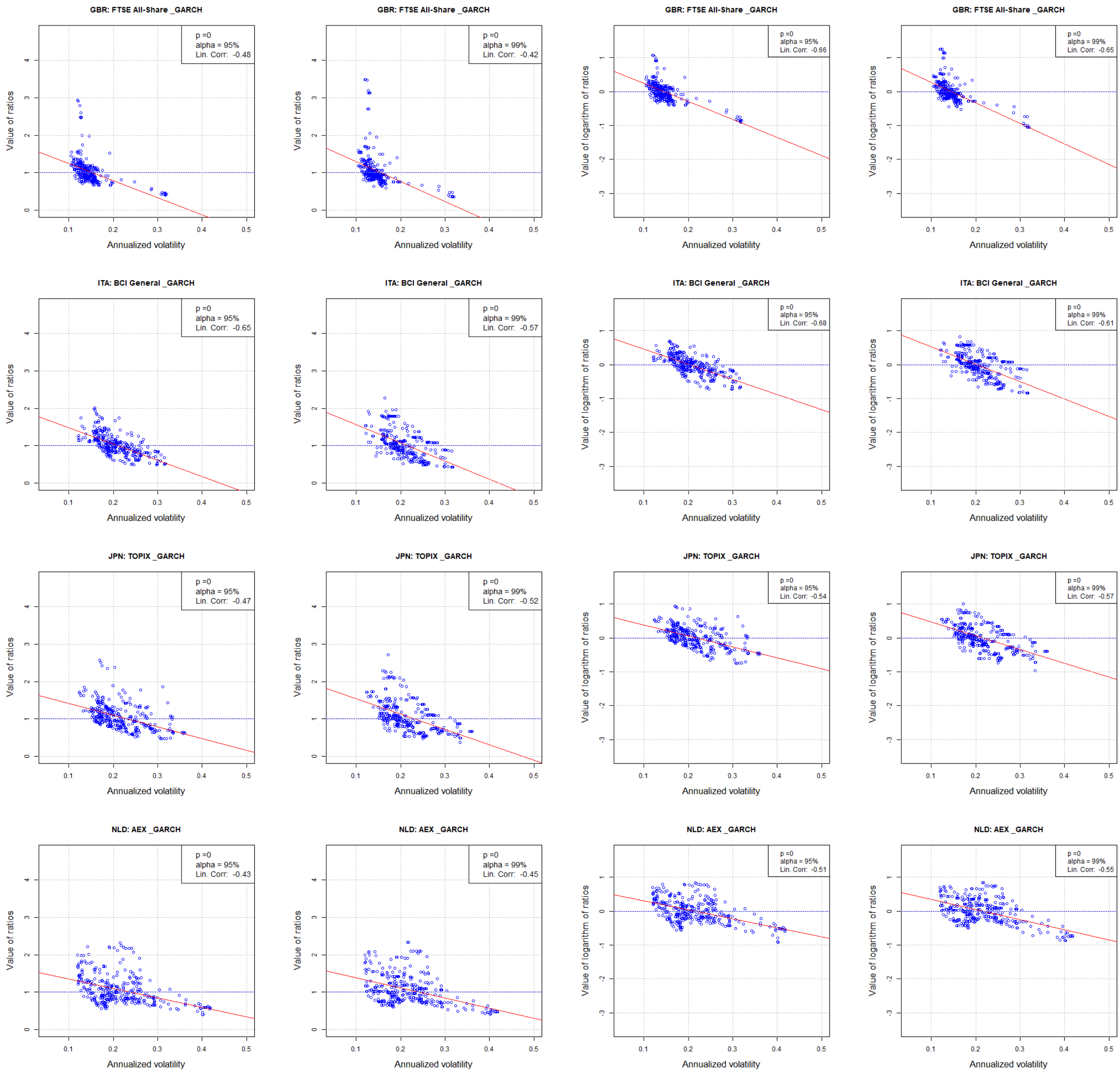

Figure A.13: SQP Ratios as a function of annualized volatility for one simulated sample path of the GARCH(1,1) - which is fitted to each index separetely - for $p = 0$ for $\alpha = 95\%, 99\%$. In each row we consider the SQP Ratios and then the logarithm of the SQP Ratios: From left to right, SQP ratio for $\alpha = 95\%, 99\%$, then logarithm of SQP ratio for $\alpha = 95\%, 99\%$. From top to bottom: GBR, ITA, JPN, NLD.

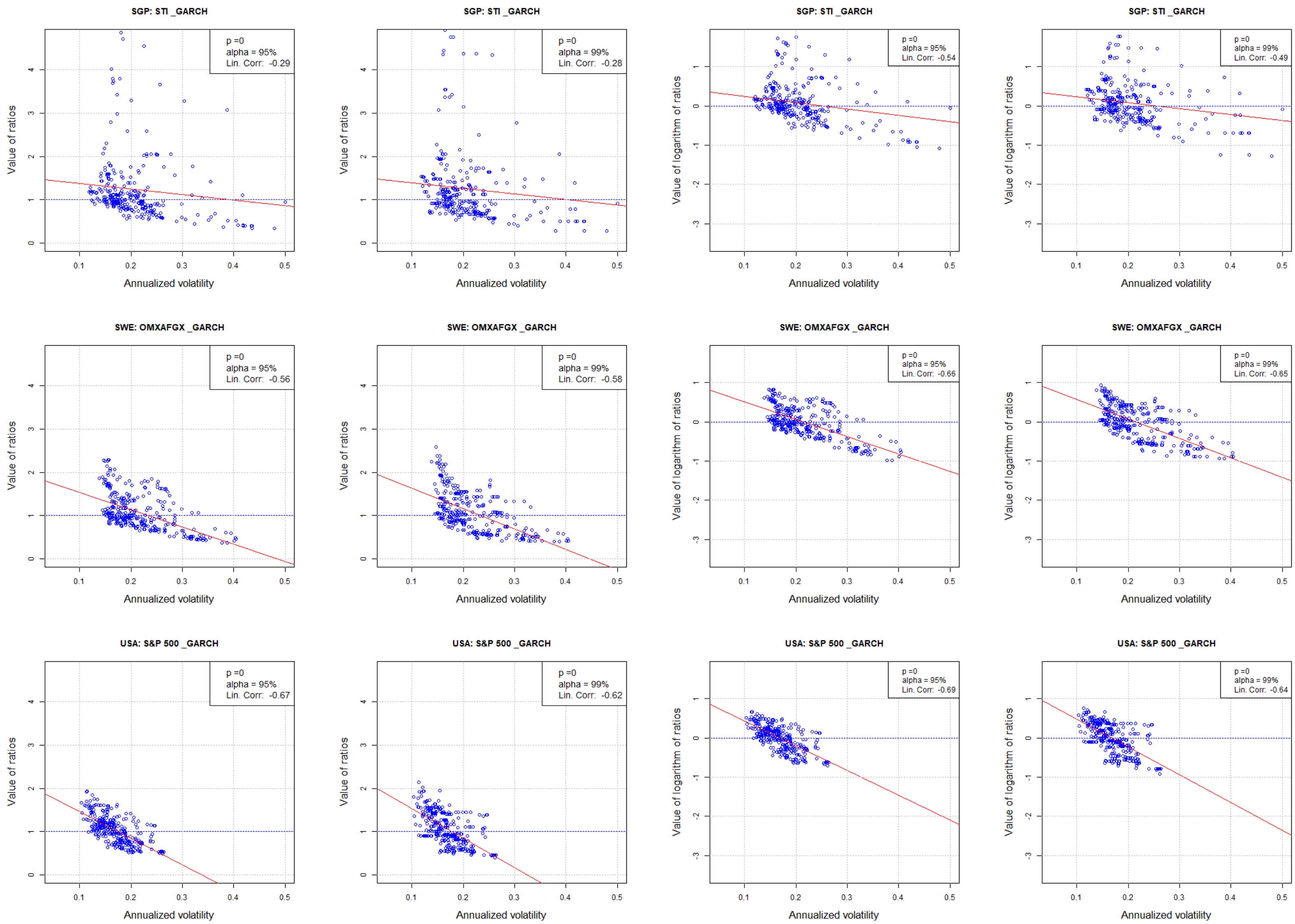

Figure A.14: SQP Ratios as a function of annualized volatility for one simulated sample path of the GARCH(1,1) - which is fitted to each index separetely - for $p = 0$ for $\alpha = 95\%, 99\%$. In each row we consider the SQP Ratios and then the logarithm of the SQP Ratios: From left to right, SQP ratio for $\alpha = 95\%, 99\%$, then logarithm of SQP ratio for $\alpha = 95\%, 99\%$. From top to bottom: SGP, SWE, USA

Appendix C: Selected Tables and Figures (3 year sample)

C.1 Tables for Rank and Linear Correlation

C.1.1 On Real Data

Table A.6: Pearson correlation between the logarithm of the SQP ratios and the volatility, for each index and for $T = 3$ years, over the whole historical sample, and for two thresholds (95 and 99%). In the last column, we present the average over all indices ± the standard deviation.

	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG (± sd)**
						$\alpha = 95\%$						
p=0	-0.38	-0.52	-0.46	-0.47	-0.47	-0.59	-0.51	-0.39	-0.53	-0.47	-0.47	**-0.48 ± 0.06**
p=0.5	-0.53	-0.56	-0.47	-0.51	-0.48	-0.66	-0.56	-0.38	-0.45	-0.57	-0.53	**-0.52 ± 0.07**
p=1	-0.66	-0.51	-0.49	-0.37	-0.42	-0.65	-0.60	-0.47	-0.38	-0.57	-0.57	**-0.52 ± 0.10**
p=2	-0.48	-0.49	-0.31	0.01	-0.23	-0.39	-0.45	-0.47	-0.26	-0.54	-0.48	**-0.37 ± 0.16**
						$\alpha = 99\%$						
p=0	-0.56	-0.54	-0.58	-0.51	-0.54	-0.62	-0.58	-0.34	-0.52	-0.53	-0.56	**-0.53 ± 0.07**
p=0.5	-0.59	-0.57	-0.49	-0.37	-0.42	-0.54	-0.60	-0.48	-0.43	-0.53	-0.62	**-0.51 ± 0.08**
p=1	-0.50	-0.52	-0.35	-0.06	-0.31	-0.44	-0.46	-0.47	-0.34	-0.53	-0.49	**-0.41 ± 0.14**
p=2	-0.48	-0.52	-0.43	-0.12	-0.38	-0.45	-0.46	-0.51	-0.42	-0.47	-0.47	**-0.43 ± 0.11**

Table A.7: Spearman rank correlation between the SQP ratios and the volatility, for each index and for $T = 3$ years, over the whole historical sample, and for two thresholds (95 and 99%). In the last column, we present the average over all indices ± the standard deviation.

	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG (± sd)**
						$\alpha = 95\%$						
p=0	-0.40	-0.53	-0.46	-0.42	-0.48	-0.62	-0.48	-0.35	-0.55	-0.52	-0.52	**-0.49 ± 0.07**
p=0.5	-0.53	-0.54	-0.48	-0.50	-0.51	-0.61	-0.56	-0.34	-0.48	-0.62	-0.54	**-0.52 ± 0.08**
p=1	-0.60	-0.51	-0.52	-0.38	-0.41	-0.46	-0.62	-0.43	-0.40	-0.64	-0.58	**-0.51 ± 0.09**
p=2	-0.49	-0.49	-0.32	-0.12	-0.16	-0.29	-0.44	-0.49	-0.26	-0.61	-0.52	**-0.38 ± 0.16**
						$\alpha = 99\%$						
p=0	-0.60	-0.54	-0.61	-0.50	-0.55	-0.53	-0.61	-0.30	-0.54	-0.58	-0.61	**-0.54 ± 0.09**
p=0.5	-0.56	-0.60	-0.51	-0.42	-0.34	-0.48	-0.63	-0.49	-0.43	-0.59	-0.67	**-0.52 ± 0.10**
p=1	-0.52	-0.54	-0.39	-0.15	-0.21	-0.35	-0.43	-0.52	-0.30	-0.59	-0.53	**-0.41 ± 0.15**
p=2	-0.48	-0.55	-0.42	-0.18	-0.27	-0.41	-0.42	-0.55	-0.42	-0.50	-0.51	**-0.43 ± 0.12**

Table A.8: Kendall rank correlation between the SQP ratios and the volatility, for each index and for $T = 3$ years, over the whole historical sample, and for two thresholds (95 and 99%). In the last column, we present the average over all indices ± the standard deviation.

	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG (± sd)**
						$\alpha = 95\%$						
p=0	-0.30	-0.37	-0.31	-0.30	-0.32	-0.45	-0.32	-0.23	-0.38	-0.36	-0.35	**-0.34 ± 0.06**
p=0.5	-0.41	-0.39	-0.32	-0.34	-0.34	-0.45	-0.39	-0.22	-0.33	-0.43	-0.37	**-0.36 ± 0.06**
p=1	-0.48	-0.36	-0.35	-0.25	-0.28	-0.33	-0.44	-0.29	-0.28	-0.43	-0.40	**-0.35 ± 0.08**
p=2	-0.36	-0.32	-0.21	-0.09	-0.09	-0.21	-0.31	-0.32	-0.16	-0.41	-0.32	**-0.25 ± 0.11**
						$\alpha = 99\%$						
p=0	-0.47	-0.38	-0.42	-0.34	-0.37	-0.38	-0.44	-0.18	-0.38	-0.41	-0.40	**-0.38 ± 0.07**
p=0.5	-0.43	-0.43	-0.34	-0.27	-0.21	-0.34	-0.45	-0.33	-0.30	-0.40	-0.46	**-0.36 ± 0.08**
p=1	-0.37	-0.37	-0.28	-0.10	-0.12	-0.24	-0.31	-0.35	-0.18	-0.38	-0.33	**-0.27 ± 0.10**
p=2	-0.34	-0.38	-0.31	-0.13	-0.17	-0.28	-0.30	-0.37	-0.26	-0.33	-0.31	**-0.29 ± 0.08**

C.1.2 On Simulated GARCH realizations

Table A.9: Pearson correlation between the logarithm of the SQP ratios and the volatility, for each index and for $T = 3$ years, using a GARCH(1,1) model (average over 1000 realizations), and for two thresholds (95 and 99%). In the last column, we present the average over all indices $\pm$ the standard deviation.

Mean for	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG ($\pm$ sd)**
						$\alpha = 95\%$						
p=0	-0.48	-0.53	-0.51	-0.52	-0.51	-0.50	-0.52	-0.51	-0.50	-0.51	-0.51	**-0.51 $\pm$ 0.10**
p=0.5	-0.59	-0.60	-0.57	-0.58	-0.56	-0.59	-0.59	-0.61	-0.57	-0.59	-0.59	**-0.58 $\pm$ 0.11**
p=1	-0.65	-0.64	-0.60	-0.62	-0.59	-0.64	-0.63	-0.65	-0.61	-0.62	-0.62	**-0.62 $\pm$ 0.12**
p=2	-0.65	-0.63	-0.60	-0.61	-0.59	-0.63	-0.63	-0.64	-0.61	-0.62	-0.62	**-0.62 $\pm$ 0.14**
						$\alpha = 99\%$						
p=0	-0.54	-0.57	-0.53	-0.55	-0.54	-0.54	-0.56	-0.56	-0.54	-0.55	-0.55	**-0.55 $\pm$ 0.11**
p=0.5	-0.59	-0.59	-0.56	-0.57	-0.55	-0.58	-0.59	-0.59	-0.57	-0.58	-0.58	**-0.58 $\pm$ 0.12**
p=1	-0.59	-0.59	-0.55	-0.57	-0.55	-0.58	-0.59	-0.59	-0.56	-0.57	-0.58	**-0.57 $\pm$ 0.14**
p=2	-0.57	-0.58	-0.54	-0.55	-0.54	-0.56	-0.57	-0.58	-0.55	-0.56	-0.56	**-0.56 $\pm$ 0.14**

Table A.10: Spearman rank correlation between the SQP ratios and the volatility, for each index and for $T = 3$ years, using a GARCH(1,1) model (average over 1000 realizations), and for two thresholds (95 and 99%). In the last column, we present the average over all indices $\pm$ the standard deviation.

Mean for	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG ($\pm$ sd)**
						$\alpha = 95\%$						
p=0	-0.49	-0.52	-0.50	-0.51	-0.50	-0.50	-0.51	-0.51	-0.49	-0.50	-0.51	**-0.50 $\pm$ 0.11**
p=0.5	-0.57	-0.58	-0.55	-0.57	-0.55	-0.57	-0.57	-0.59	-0.56	-0.56	-0.57	**-0.57 $\pm$ 0.11**
p=1	-0.62	-0.61	-0.58	-0.59	-0.57	-0.61	-0.60	-0.62	-0.59	-0.59	-0.60	**-0.60 $\pm$ 0.12**
p=2	-0.62	-0.61	-0.57	-0.59	-0.57	-0.60	-0.60	-0.61	-0.59	-0.59	-0.60	**-0.59 $\pm$ 0.14**
						$\alpha = 99\%$						
p=0	-0.52	-0.54	-0.52	-0.53	-0.52	-0.52	-0.54	-0.54	-0.52	-0.53	-0.53	**-0.53 $\pm$ 0.12**
p=0.5	-0.56	-0.57	-0.53	-0.55	-0.53	-0.56	-0.56	-0.57	-0.54	-0.55	-0.56	**-0.55 $\pm$ 0.12**
p=1	-0.56	-0.56	-0.53	-0.54	-0.53	-0.55	-0.56	-0.56	-0.54	-0.54	-0.55	**-0.55 $\pm$ 0.13**
p=2	-0.55	-0.55	-0.52	-0.53	-0.52	-0.54	-0.55	-0.55	-0.53	-0.53	-0.54	**-0.54 $\pm$ 0.14**

Table A.11: Kendall rank correlation between the SQP ratios and the volatility, for each index and for $T = 3$ years, using a GARCH(1,1) model (average over 1000 realizations), and for two thresholds (95 and 99%). In the last column, we present the average over all indices $\pm$ the standard deviation.

Mean for	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG ($\pm$ sd)**
						$\alpha = 95\%$						
p=0	-0.35	-0.37	-0.36	-0.36	-0.36	-0.35	-0.37	-0.37	-0.35	-0.36	-0.36	**-0.36 $\pm$ 0.08**
p=0.5	-0.41	-0.42	-0.39	-0.41	-0.39	-0.41	-0.41	-0.42	-0.40	-0.41	-0.41	**-0.41 $\pm$ 0.09**
p=1	-0.45	-0.44	-0.41	-0.43	-0.41	-0.44	-0.44	-0.45	-0.42	-0.43	-0.43	**-0.43 $\pm$ 0.09**
p=2	-0.45	-0.44	-0.41	-0.42	-0.41	-0.43	-0.43	-0.44	-0.42	-0.42	-0.43	**-0.43 $\pm$ 0.11**
						$\alpha = 99\%$						
p=0	-0.37	-0.39	-0.37	-0.38	-0.37	-0.38	-0.39	-0.39	-0.37	-0.38	-0.38	**-0.38 $\pm$ 0.09**
p=0.5	-0.40	-0.41	-0.38	-0.39	-0.38	-0.40	-0.40	-0.41	-0.39	-0.39	-0.40	**-0.40 $\pm$ 0.10**
p=1	-0.40	-0.40	-0.38	-0.39	-0.38	-0.39	-0.40	-0.41	-0.38	-0.39	-0.40	**-0.39 $\pm$ 0.10**
p=2	-0.39	-0.40	-0.37	-0.38	-0.37	-0.38	-0.39	-0.40	-0.37	-0.38	-0.39	**-0.38 $\pm$ 0.11**

C.2 Figures for the S&P 500

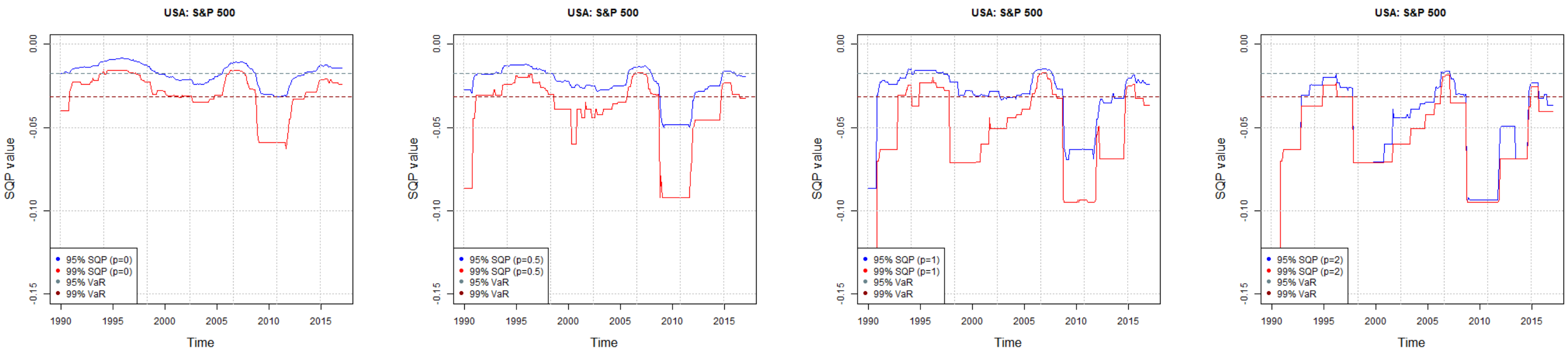

Figure A.15: SQP rolling every month with $T = 3$ year, and thresholds $\alpha = 95\%$ and 99% and different values of p using the S&P 500 index. From left to right: $p = 0$, $p = 0.5$, $p = 1$ and $p = 2$. Dashed lines correspond to VaR(α) computed over the whole sample.

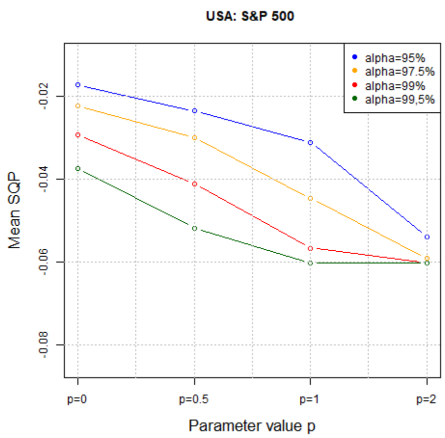

Figure A.16
Average over the whole sample of rolling-window SQPs with sample size $T = 3y$ (S&P 500) as a function of the power p for various thresholds α.

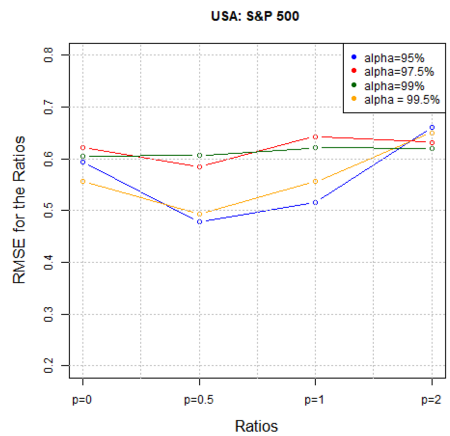

Figure A.17
RMSE (defined in (3.2)) of the SQP Ratios $R_{p,\alpha,3}(t)$ (i.e. on a 3-year sample) taking $p = 0, 0.5, 1, 2$ and two thresholds $\alpha = 95\%$ (in blue), 99% (in red), on the S&P 500

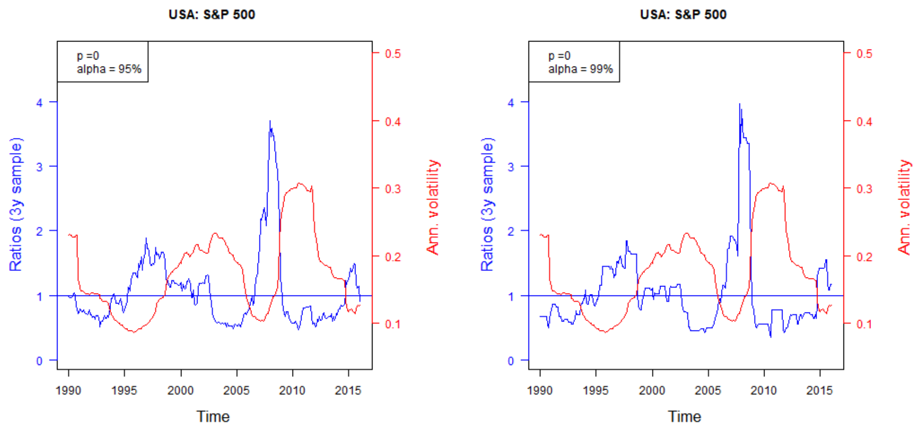

Figure A.18: SQP Ratios and annualized volatility time series computed over 3 years ($T = 3y$) for $\alpha = 95\%$ (left) and $\alpha = 99\%$ (right)

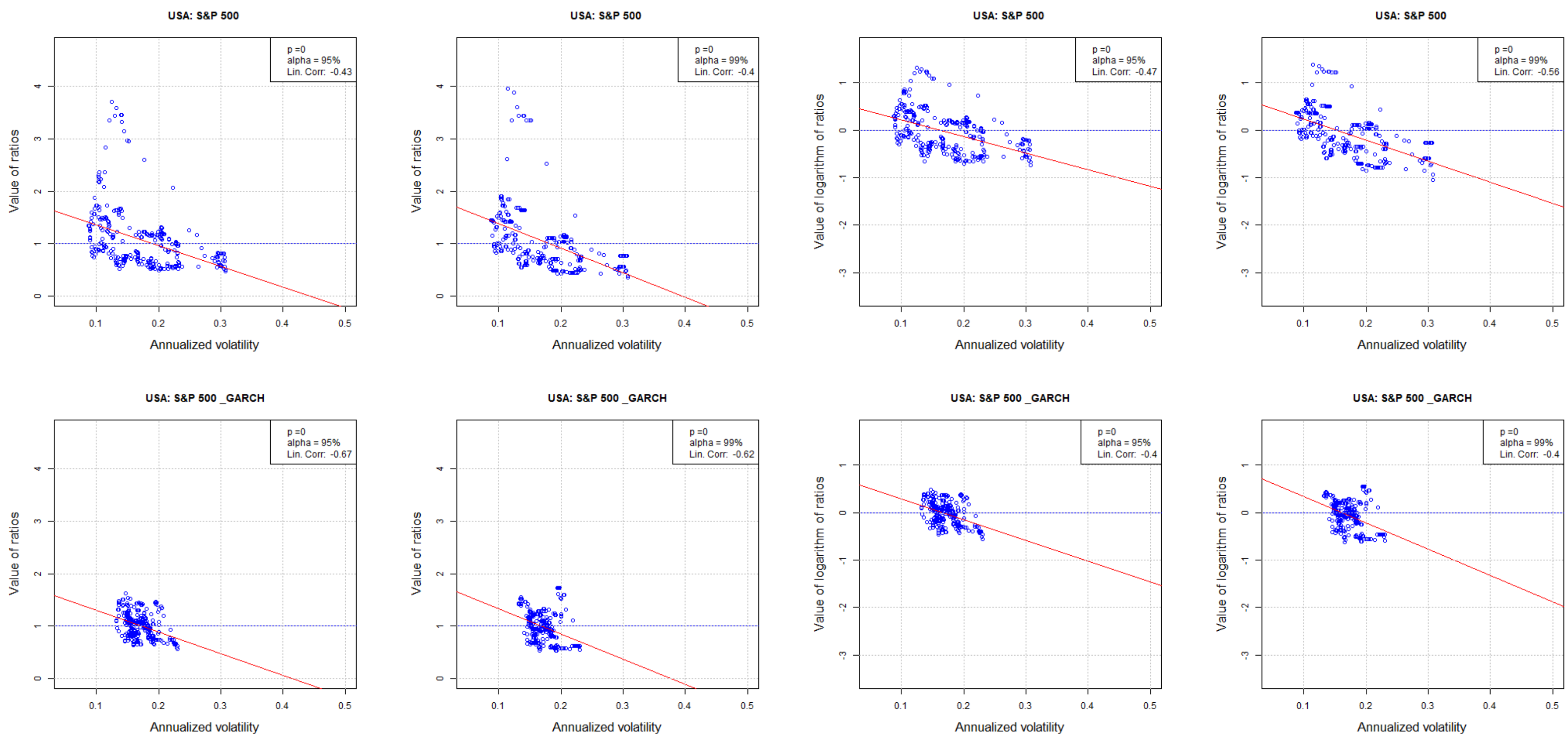

Figure A.19: SQP Ratios as a function of annualized volatility for $p = 0$ for $\alpha = 95\%, 99\%$. In each row we consider the SQP Ratios and then the logarithm of the SQP Ratios: From left to right, SQP ratio for $\alpha = 95\%, 99\%$, then logarithm of SQP ratio for $\alpha = 95\%, 99\%$. From top to bottom: On real data, on a realization of a GARCH(1,1) simulation.

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